MATH 110 Spring 2019 midterm test #1 solutions.

1. (10pp.) Let v_1, v_2, …, v_n be a basis of a vector space V. Determine, with proof, the dimension of $\operatorname{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1)$.

Solution: Note that all vectors on the list sum up to the zero vector:

$$(v_1 - v_2) + (v_2 - v_3) + \cdots + (v_n - v_1) = 0.$$

That means their nontrivial combination (with all coefficients equal to one) is zero. Hence this list of vectors is linearly dependent.

By 2.31, this list can be reduced to a basis of the vector space $\operatorname{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1)$. (The latter is a subspace of V by 2.7, hence a vector space in its own right.) As the original list is linearly dependent, that reduction to a basis has to remove at least one vector from the list, leaving us with $n - 1$ or fewer vectors. Thus

$$\dim \operatorname{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1) \leq n - 1.$$

On the other hand, consider

$$W := \operatorname{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1) + \operatorname{span}(v_1).$$

Note that $v_1 \in W$ and that

$$v_j = -v_1 - \sum_{k=1}^{j-1} (v_k - v_{k+1}) \in W \qquad \text{for all} \quad j = 2, \ldots, n$$

Since $(v_1, v_2, \ldots, v_n)$ is a basis for V by assumption of the problem, this implies $V \subseteq W$ by 2.7, hence $V = W$ since V is the entire space. The space $V = W$ is thus n-dimensional.

Now, by 2.43,

$$\begin{aligned} n &= \dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) + \dim \operatorname{span}(v_1) - \dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) \cap \operatorname{span}(v_1) \\ &\leq \dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) + \dim \operatorname{span}(v_1) = \dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) + 1. \end{aligned}$$

The last equality here is due to the 1-dimensionality of $\operatorname{span}(v_1)$, the span of a single non-zero vector. Subtracting 1 from both sides of the last line, we get

$$\dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) \geq n - 1.$$

Thus, $\dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) \leq n - 1$ and $\dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) \geq n - 1$, so $\dim \operatorname{span}(v_1 - v_2, \ldots, v_n - v_1) = n - 1$.

Answer: $n - 1$.

2. (10pp.) Let $V = \mathbb{R}^4$, let $W_1 = \{(x_1, x_2, x_3, x_4) : x_3 + x_4 = 0 \in \mathbb{R}\}$, and let $W_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$.

(a) Prove that W_1 and W_2 are subspaces of V.

Proof: The zero vector $(0, 0, 0, 0)$ is in W_1 since the sum of its third and fourth components is zero. Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be elements of W_1, and let $\alpha, \beta \in \mathbb{R}$. Then $x_3 + x_4 = y_3 + y_4 = 0$ by the definition of W_1. Hence

$$(\alpha x_3 + \beta y_3) + (\alpha x_4 + \beta y_4) = \alpha(x_3 + y_3) + \beta(x_4 + y_4) = 0.$$

So $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4) \in W_1$. Taking $\alpha = \beta = 1$, this shows that W is closed under addition. Taking $\beta = 0$, this shows that W is closed under scalar multiplication. By the Subspace Test 1.34, W_1 is a subspace of V.

The proof for W_2 is analogous: $0 \in W_2$ and $\alpha x + \beta y \in W_2$ whenever x, $y \in W_2$ and α, $\beta \in \mathbb{R}$ because $\sum_{j=1}^4 (\alpha x_j + \beta y_j) = \alpha \sum_{j=1}^4 x_j + \beta \sum_{j=1}^4 y_j = 0$. So W_2 is a subspace too.

(b) Is the sum $W_1 + W_2$ direct? Explain why or why not.

Solution: For the sum $W_1 + W_2$ to be direct, we must have $W_1 \cap W_2 = \{0\}$ by 1.45. However, the nonzero vector $(0, 0, 1, -1)$ belongs to both W_1 and W_2 since it satisfies both conditions defining those subspaces. So the sum $W_1 + W_2$ is not direct. **Answer:** no.

(c) Determine $\dim(W_1 + W_2)$.

Solution: Let us check that $W_1 + W_2 = V$. Given an arbitrary vector (x_1, x_2, x_3, x_4), we can rewrite it as

$$(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, 0, 0, 0) + (-x_2 - x_3 - x_4, x_2, x_3, x_4).$$

By the defintion of W_1, the first summand is in W_1 since its third and fourth components sum up to zero. By the definition of W_2, the second summand is in is in W_2 since all its four coordinates sum up to zero.

This shows $V \subseteq W_1 + W_2$. Since V is the entire space, we also have $V \supseteq W_1 + W_2$. Hence $V = W_1 + W_2$, and $\dim W_1 + W_2 = 4$. **Answer:** 4.

3. (10pp.) Let V be the vector space of all real-valued infinitely differentiable functions over $\mathbb{R}$ and let $n \in \mathbb{N}$. Prove or disprove: the list of maps $T_0, \ldots, T_n \in \mathcal{L}(V, V)$ where $T_j : f(x) \mapsto x^j f^{(j)}(x)$ for $j = 0, \ldots, n$, is linearly independent.

Solution: Assume there exist α_j, $j = 0, \ldots, n$ such that

$$\sum_{j=0}^{n} (T_j f)(x) = \sum_{j=0}^{n} \alpha_j x^j f^{(j)}(x) \equiv 0 \tag{1}$$

for all infinitely differentiable functions f over $\mathbb{R}$. We will prove that all $\alpha_j = 0$, $j = 0, \ldots, n$ by (strong) induction.

Induction base: $j = 0$. Take $f_0(x) = 1$. Then $f_0^{(k)} = 0$ for $k \geq 1$, hence

$$\sum_{j=0}^{n} \alpha_j x^j f_0^{(j)}(x) \equiv \alpha_0 \equiv 0.$$

This shows $\alpha_0 = 0$.

Induction hypothesis: For a given j, suppose that we have already proved that (1) implies that $\alpha_i = 0$ for all $0 \leq i < j$.

Induction step: Take $f_j(x) = x^j$. Then $f_j^{(k)} \equiv 0$ for $k > j$, $f_j^{(j)} \equiv j!$, and $\alpha_i = 0$ for all $i < j$ by the inductive hypothesis. Hence

$$0 \equiv \sum_{k=0}^{n} \alpha_k x^k f_j^{(k)}(x) \equiv \alpha_j x^j j!$$

which implies $\alpha_j = 0$. This completes the inductive step.

Conclusion: $\alpha_j = 0$ for all $j = 0, \ldots, n$, hence the list of maps is linearly independent.

Answer: linearly independent.

4. (10pp.) Consider the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}) : f(x) \mapsto f(x+1) - f(x)$.

(a) Prove that T is linear.

Proof: Let f, $g \in \mathcal{P}_2(\mathbb{R})$ and let α, $\beta \in \mathbb{R}$. We get

$$\begin{aligned} T(\alpha f + \beta g)(x) &= (\alpha f + \beta g)(x+1) - (\alpha f + \beta g)(x) \\ &= \alpha(f(x+1) - f(x)) + \beta(g(x+1) - g(x)) \\ &= \alpha(Tf)(x) + \beta(Tg)(x). \end{aligned}$$

This shows T preserves linear combinations of vectors in its domains, hence T is linear.

(b) Is T surjective?

Solution: Note that $x^2 \notin \operatorname{range} T$. Indeed, suppose $x^2 = (Tf)(x)$ for some $f \in \mathcal{P}_2(\mathbb{R})$. Say, $f(x) = ax^2 + bx + c$. Then

$$(Tf)(x) = f(x+1) - f(x) = a((x+1)^2 - x^2) + b((x+1) - x) + c(1-1) = a(2x+1) + b. \quad (2)$$

The polynomial we obtain in the right-hand side has degree at most 1, which cannot equal x^2. Hence the range of T is not the entire co-domain $\mathcal{P}_2(\mathbb{R})$, which means T is not surjective.

Answer: no.

(c) Find the dimension of $\operatorname{range} T$.

Solution: By taking $a = 1/2$ and $b = -1/2$, and by taking $a = 0$ and $b = 1$ in (2), we see that x and 1 are in the range of T. Since $\operatorname{range} T$ is a subspace of its co-domain by 3.19, this implies $\operatorname{range} T \supseteq \operatorname{span}(1, x) = \mathcal{P}_1(\mathbb{R})$. We know that $\dim \mathcal{P}_1(\mathbb{R}) = 2$ by 2.28(g), hence $\dim \operatorname{range} T \geq 2$. By the result of (b), $\operatorname{range} T$ is a *proper* subspace of its 3-dimensional co-domain $\mathcal{P}_2(\mathbb{R})$ (the latter dimensionality again by 2.28(g)).

Hence, by 2.34, $\operatorname{range} T$ has a nonzero direct complement within its co-domain $\mathcal{P}_2(\mathbb{R})$ therefore $\dim \operatorname{range} T$ is strictly less than $\dim \mathcal{P}_2(\mathbb{R}) = 3$.

Thus $\dim \operatorname{range} T \geq 2$ and $\dim \operatorname{range} T \leq 2$, so $\dim \operatorname{range} T = 2$.

Answer: 2.