Math 110, Spring 2019.

Homework 9 solutions.

Prob 1. Let V be a complex vector space and let $T \in \mathcal{L}(V)$ satisfy $(T-2I)(T+4I)(T-7I) = 0$. What possible values can $\lambda \in \mathbb{C}$ take for it to be an eigenvalue of T?

Solution. Suppose λ and v are an eigenvalue-eigenvector pair for T, i.e., $Tv = \lambda v$ and $v \neq 0$. Then

$$0 = (T-2I)(T+4I)(T-7I)v = (\lambda-2)(\lambda+4)(\lambda-7)v,$$

hence $(\lambda-2)(\lambda+4)(\lambda-7) = 0$, hence $\lambda \in \{2, -4, 7\}$.

Answer: The possible eigenvalues are 2, -4, and 7.

Prob 2. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues.

(a) Prove that every subspace of V invariant under T is either zero or infinite-dimensional.

Proof. We will show the contrapositive, i.e., that a finite-dimensional nonzero subspace cannot be invariant under the action of T. Indeed, suppose $U \neq \{0\}$ is finite-dimensional and invariant under the action of T. Then the restriction of T to U is well-defined (since vectors from U are mapped to vectors from U). Therefore, $T|_U \in \mathcal{L}(U)$. By 5.21, $T|_U$ has an eigenvalue λ (and an associated eigenvector $u(\neq 0) \in U$. But then

$$Tu = T|_U u = \lambda u,$$

that is, λ is an eigenvalue of T, contrary to the assumption of this problem. Contradiction!

(b) Give an example of such an operator T on $V := \mathbb{C}^\infty$ with a T-invariant nonzero proper subspace.

Example. Consider the shirt operator $T \in \mathcal{L}(V)$ on $V = \mathbb{C}^\infty$:

$$T : \ (z_1, z_2, z_3, \ldots) \mapsto (0, z_1, z_2, z_3, \ldots).$$

Suppose λ and $z = (z_1, z_2, z_3, \ldots)(\neq 0)$ is an eigenpair for T. This means

$$\lambda z_1 = 0, \quad \lambda z_2 = z_1, \quad \lambda z_3 = z_2, \ldots.$$

If $\lambda = 0$, then the first equation is satisfied, and the subsequent equations imply $z_j = 0$ for all $j \in \mathbb{N}$. If $\lambda \neq 0$, then the first equation implies $z_1 = 0$, the second implies $z_2 = 0$, etc., so again all $z_j = 0$.

This shows that T has no eigenvalues. On the other hand, the subspace from 1.35(e)

$$\{(z_1, z_2, z_3, \ldots) \in V : \lim_{n\to\infty} z_n = 0\}$$

is invariant under T since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_{n-1}$ whenever the limit in the left-hand side exists.

Prob 3. Let V be a finite-dimensional complex vector space. Prove that $T \in \mathcal{L}(V)$ is diagonalizable if and only if, for all $\lambda \in \mathbb{C}$,

$$\text{null}\,(T - \lambda I) \oplus \text{range}\,(T - \lambda I) = V. \tag{1}$$

Proof. Suppose T is diagonalizable and $v \in V$. We will show that, for any $\lambda \in \mathbb{C}$, the condition $v \in \text{null}\,(T - \lambda I)^2$ implies $v \in \text{null}\,(T - \lambda I)$. Indeed, since T is diagonalizable, v can be written as a linear combination of eigenvectors of T corresponding to distinct eigenvalues (a group of summands consisting of eigenvectors corresponding to the same eigenvalue will be considered a single eigenvector): $v = \sum_{j=1}^{k} \alpha_j v_j$ where $Tv_j = \lambda_j v_j$ for all j, and the λ_js are distinct. Then

$$(T - \lambda I)v = \sum_{j=1}^{k} \alpha_j(\lambda_j - \lambda)v_j \quad \text{and} \quad (T - \lambda I)^2 v = \sum_{j=1}^{k} \alpha_j(\lambda_j - \lambda)^2 v_j.$$

By 5.10, eigenvectors corresponding to distinct eigenvalues are linearly independent, so $v \in \text{null}\,(T - \lambda I)^2$ implies $\alpha_j(\lambda_j - \lambda)^2 = 0$ for all j, which, in turn, implies $\alpha_j(\lambda_j - \lambda) = 0$ for all j. Hence $v \in \text{null}\,(T - \lambda I)$.

This now implies that $\text{null}\,(T - \lambda I) \cap \text{range}\,(T - \lambda I) = \{0\}$ for any $\lambda \in \mathbb{C}$ because any vector w simultaneously in $\text{null}\,(T - \lambda I)$ and $\text{range}\,(T - \lambda I)$ must have the form $w = (T - \lambda I)v$ where $(T - \lambda)^2 v = 0$, i.e., $v \in \text{null}\,(T - \lambda I)^2$. But then $w = (T - \lambda)v = 0$, as we just proved. So $\text{null}\,(T - \lambda I) \cap \text{range}\,(T - \lambda I) = \{0\}$, hence the sum $\text{null}\,(T - \lambda I) + \text{range}\,(T - \lambda I)$ is direct. By the Rank-Nullity Theorem, $\dim \text{null}\,(T - \lambda I) + \dim \text{range}\,(T - \lambda I) = \dim V$, so this direct sum equals V by 2.43. Hence (1) holds for all $\lambda \in \mathbb{C}$.

Now suppose (1) holds for all $\lambda \in \mathbb{C}$. Show that T is diagonalizable by induction on $\dim V$. The result is true for $\dim V = 1$ (induction base) because every operator on a 1-dimensional space is automatically in diagonal form with respect to any basis. Now assume the diagonalization property holds on all complex vector spaces of dimension less than $\dim V$. By 5.21, there exists an eigenvalue $\lambda \in \mathbb{C}$ of T. Because $T - \lambda I$ is not injective, it is also not surjective. Thus $\text{range}\,(T - \lambda I)$ is a subspace of V of dimension less than $\dim V$. Apply the induction hypothesis to the operator $T|_{\text{range}\,(T - \lambda I)}$ to conclude there is a basis of $\text{range}\,(T - \lambda I)$ consisting of eigenvectors of T. Adjoin this basis to any basis of $\text{null}\,(T - \lambda I)$, getting a basis of V because of (1). Thus T is diagonalizable.

Prob 4. Determine whether or not the function taking the pair $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1y_2 + 2x_2y_3 + 3x_3y_1$ is an inner product.

Solution. This is not an inner product because, say, the pair of vectors $(0, -1, 1)$, $(0, 1, 0)$ is mapped to zero but the 'flipped' pair $(0, 1, 0)$, $(0, -1, 1)$ is mapped to 1, so the symmetry condition fails.

Prob 5. Use the dot product to show that the diagonals of a rhombus are perpendicular to each other.

Proof. Consider the plane $V := \mathbb{R}^2$, and let u be the vector along one side of the rhombus and v the vector along the other. In a rhombus, the lenghts of these vectors are the same, i.e., $\|u\| = \|v\|$. Then the vectors along the diagonals are $u + v$ and $u - v$. Consider the dot product of these vectors:

$$\langle u + v, u - v\rangle = \langle u, u\rangle + \langle u, v\rangle - \langle v, u\rangle - \langle u, u\rangle = \|u\|^2 - \|v\|^2 = 0.$$

So the vectors $u + v$ and $u - v$, and hence the diagonals of the rhombus, are indeed orthogonal.

Prob 6. Prove that, for all complex numbers a_j, b_j, $j = 1, \dots, n$, the following inequality holds:

$$\Big|\sum_{j=1}^{n} a_j \overline{b_j}\Big|^2 \le \left(\sum_{j=1}^{n} j\,|a_j|^2\right)\left(\sum_{j=1}^{n} \frac{|b_j|^2}{j}\right).$$

Proof. Consider the space $\mathbb{C}^n$ with the standard inner product

$$\langle x, y\rangle := \sum_{j=1}^{n} x_j \overline{y_j}.$$

Given the complex numbers a_j, b_j, $j = 1, \dots, n$, let

$$x := (a_1, \sqrt{2}a_2, \sqrt{3}a_3, \dots, \sqrt{n}a_n), \qquad y := \left(b_1, \frac{b_2}{\sqrt{2}}, \frac{b_3}{\sqrt{3}}, \dots, \frac{b_n}{\sqrt{n}}\right)$$

and apply the Cauchy-Schwarz inequality. We get

$$\Big|\sum_{j=1}^{n} a_j \overline{b_j}\Big|^2 = |\langle x, y\rangle|^2 \le \|x\|^2 \cdot \|y\|^2 = \left(\sum_{j=1}^{n} j\,|a_j|^2\right)\left(\sum_{j=1}^{n} \frac{|b_j|^2}{j}\right).$$