Math 110, Spring 2019.

Homework 8 solutions.

Prob 1. Let T, $S \in \mathcal{L}(V)$ be such that $TS = ST$. Show that $\operatorname{range} T$ and $\operatorname{null} T$ are invariant under S.
Solution: Let us begin by showing that range(T) is S-invariant. This entails showing that for any $v \in V$ we have that for any $v \in V$ we have that $S(T(v)) = T(v')$ for some $v' \in V$. Note though that $S(T(v)) = T(S(v))$ since S and T commute, so we can merely take $v' = S(v)$.

Let us now show that null(T) is S-invariant. Again, this is equivalent to showing a more concrete statement: if $T(v) = 0$ then $T(S(v)) = 0$. But, note $T(S(v)) = S(T(v)) = S(0) = 0$ where the first step used the commtuativity of T and S.

Prob 2. Let $S, T \in \mathcal{L}(V)$ and suppose S is invertible. Prove that, for any polynomial $p \in \mathcal{P}(\mathbb{F})$,

$$p(STS^{-1}) = S\,p(T)\,S^{-1}.$$

Solution: Let us note since polynomials are comprised of sums of terms of the form $a_n x^n$ it suffices to show the following three things:

1. If A and B are operators then $S(A+B)S^{-1} = SAS^{-1} + BSB^{-1}$.
2. If A is an operator and λ is a scalar then $\lambda SAS^{-1} = S(\lambda A)S^{-1}$.
3. If A is an operator and $n \in \mathbb{N}$ then $(SAS^{-1})^n = SA^nS^{-1}$.

The first property follows immediately from the distributivity property of function composition and addition. The second follows from the linearity of S. To see the third we proceed by induction. For $n = 1$ this is clear. Assume the result is true for n. Then,

$$\begin{aligned}
(SAS^{-1})^{n+1} &= (SAS^{-1})(SAS^{-1})^n \\
&= (SAS^{-1})(SA^nS^{-1} \\
&= SAS^{-1}SA^nS^{-1} \\
&= SAA^nS^{-1} \\
&= SA^{n+1}S^{-1}
\end{aligned} \tag{1}$$

from where the conclusion follows.

Prob 3. Let v be an eigenvector of $T \in \mathcal{L}(V)$ with eigenvalue λ. Show that

$$(T^3 + 3T^2 - 4T + I)v = (\lambda^3 + 3\lambda^2 - 4\lambda + 1)v.$$

How does this observation generalize?
Solution: Let us prove the following generalization, since it requires no extra work. Namely, let $p(x)$ be a polynomial. Then, for v, T, and λ as in the problem statement the equality $p(T)v = p(\lambda)v$ holds. Again, using the same reasoning in the previous problem it suffices to show that $aT^n v = a\lambda^n v$ since ever polynomial is expressed as a sum of such terms. Again, to be rigorous, we proceed by induction. For $n = 1$ the assertion is clear. If the result is true for n then

$$\begin{aligned} aT^{n+1}v &= aT(T^n(v)) \\ &= T(aT^n(v)) \\ &= T(\lambda^n v) \\ &= \lambda^n T(v) \\ &= \lambda^n \lambda v \\ &= \lambda^{n+1} \end{aligned} \tag{2}$$

from where the conclusion follows.

Prob 4. Let V be a finite-dimensional real vector space and let $T \in \mathcal{L}(V)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\lambda) := \dim \text{range}\,(T - \lambda I).$$

Which condition on T is equivalent to f being a continuous function?
Solution:

Remark: We should assume that $V \neq 0$. If V is the zero space then there is only one operator T (the zero map) and the function f is continuous. But, we discuss eigenvalues below, and eigenvalues on the zero space are weird (e.g. if we require the existence of a non-zero eigenvector for an eigenvalue, then the claim that T an operator on a finite-dimensional $\mathbb{C}$-space V has an eigenvalue is only true if $V \neq 0$). So, we assume that $V \neq 0$ in the following.

Let us begin by noting that f's range lies inside of $\{0, 1, \ldots, \dim V\}$. Indeed, for any λ since $T - \lambda I$ is an operator on V we have that range$(T - \lambda I)$ is a subspace of V, and so has dimension in the claimed range. We claim that this observation shows that f is continuous if and only if f is constant. Indeed, if f is constant it's certainly continuous. Conversely, suppose that f is continuous. If $\lambda_0, \lambda_1 \in \mathbb{R}$ are such that $f(\lambda_0) \neq f(\lambda_1)$ (assume without loss of generalith that $f(\lambda_0) < f(\lambda_1)$ then by the Intermediate Value Theorem we have that f takes all values in $[f(\lambda_0), f(\lambda_1)]$. But, since between any two integers there is a non-integer we see that this implies that f takes non-integer values, which is preposterous. Thus, f is necessarily constant.

We now seek conditions on T that guarantee that f is constant. Let $\lambda_0 \in \mathbb{R}$ be a non-eigenvalue of T. Note that such a λ_0 exists since T has only finitely many eigenvalues and $\mathbb{R}$ contains infinitely many elements. Note then that, by definition, $\text{Null}(T - \lambda_0 I) = 0$. By the Rank-Nullity theorem this implies that

$$f(\lambda_0) = \dim \text{range}(T - \lambda_0 I) = \dim V \tag{3}$$

Thus, if f is constant we see that f must be the constant function $f(\lambda) = \dim V$. Note though that $f(\lambda) < \dim V$ if and only if $\text{Null}(T - \lambda I) \neq 0$ if and only if λ is a (real) eigenvalue of T. In particular, we see that f is constant if and only if it has no (real) eigenvalues.

EDIT EDIT: The below is not necessary. This is already proved in your book in either 5.9 or 5.13 depending on your edition.
EDIT: To see the claim above that T has only finitely many eigenvalues, we can proceed as follows. Let $\lambda_1, \ldots \lambda_k$ be distinct eigenvalues of T and let $u_1, \ldots, u_k$ be associated eigenvectors. We claim that $u_1, \ldots, u_k$ are independent. To see this, suppose first that $k = 2$. Then, if u_1 and u_2 are dependent we'd have that $u_1 = \alpha u_2$ for some $\alpha \neq 0$. Note then that $T(u_1) = \lambda_1 u_1$ but also $T(u_1) = T(\alpha u_2) = \alpha \lambda_2 u_2 = \lambda_2 u_1$. Since $\lambda_1 \neq \lambda_2$ this is a contradiciton. For general k we reduce to the $k = 2$ case. Indeed, suppose that

$$u_1 = \alpha_2 u_2 + \cdots + \alpha_k u_k \tag{4}$$

evidently with at lest one of the $\alpha_i \neq 0$. Applying T to both sides gives

$$\lambda_1 u_1 = \lambda_2 \alpha_2 u_2 + \cdots + \lambda_k \alpha_k u_k \tag{5}$$

Multiplying (4) by λ_1 and subtracting it form (5) gives

$$0 = \alpha_2(\lambda_2 - \lambda_1)u_2 + \cdots + \alpha_k(\lambda_k - \lambda_1)u_k \tag{6}$$

Note that since $\lambda_1 \neq \lambda_i$ for any $i > 1$ we have created a dependence on $u_2, \ldots, u_k$. Continuing this process will eventually give you a dependence on u_{k-1}, u_k which is impossible. Note then that implies that T can have at most $\dim V$ distinct eigenvalues, since otherwise we could produce a list of independent vectors in V of size larger that $\dim V$ which is impossible.

Prob 5. Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$ is diagonalizable, and all eigenvalues of T are strictly below 1 in absolute value. Given $\varphi \in V'$ and $v \in V$, what is $\lim_{n\to\infty} \varphi(T^n v)$?
Solution: Since T is diagonalizable we can fix an eigenbasis $\{v_1, \ldots, v_n\}$ of V and let λ_i be the eigenvalue associated to v_i (note that it's possible that $\lambda_i = \lambda_j$ for $i \neq j$, but this doesn't matter). Note then that if $v = \alpha_1 v_1 + \cdots + \alpha_n v_n \in V$ then

$$\begin{aligned} T^m(v) &= T^m\left(\sum_{i=1}^{m} \alpha_i v_i\right) \\ &= \sum_{i=1}^{n} \alpha_i T^m(v_i) \\ &= \sum_{i=1}^{n} \alpha_i \lambda_i^m \end{aligned} \tag{7}$$

where we have used the claim that $T^m(v_i) = \lambda^m v_i$ which follows form Problem 3. Note then that

$$\begin{aligned} \varphi(T^m(v)) &= \varphi\left(\sum_{i=1}^{n} \alpha_i \lambda_i^m\right) \\ &= \sum_{i=1}^{n} \alpha_i \lambda_i^m \varphi(v_i) \end{aligned} \tag{8}$$

So, to prove that $\lim\limits_{m\to\infty} \varphi(T^m(v)) = 0$ we proceed as follows. Recall from basic calculus/analysis that if (a_m) and (b_m) are sequences of complex numbers such that $\lim\limits_{m\to\infty} a_m$ and $\lim\limits_{m\to\infty} b_m$ exists then, for any scalars $\alpha, \beta \in \mathbb{C}$ we have that $\lim\limits_{m\to\infty}(\alpha a_m + \beta b_m)$ exists and, moreover, that

$$\lim_{m\to\infty}(\alpha a_m + \beta b_m) = \alpha(\lim_{m\to\infty} a_m) + \beta(\lim_{m\to\infty} b_m) \tag{9}$$

From this, it's clear from (8) that to show $\lim\limits_{m\to\infty} \varphi(T^m(v)) = 0$ it suffices to show that for each i we have that $\lim\limits_{m\to\infty} \lambda_i^m = 0$ (note that $\varphi(v_i)$ is some fixed scalar!). From basic calculus/analysis it's also well-known that if (a_m) is a sequence of complex numbers then $\lim\limits_{m\to\infty} a_m = 0$ if and only if $\lim\limits_{m\to\infty} |a_m| = 0$. Thus, it suffices to justify why $\lim\limits_{m\to\infty} |\lambda_i^m| = 0$. But, $|\lambda_i^m| = |\lambda_i|^m$. Since $|\lambda_i| < 1$ the limit $\lim\limits_{m\to\infty} |\lambda_i|^m = 0$ clearly holds.