Math 110, Spring 2019.

Homework 7 solutions.

Prob 1. Suppose that V is a finite-dimensional vector space with subspaces U and W. Show that

$$(U \cap W)^\circ = U^\circ + W^\circ \tag{1}$$

Solution: Since $U^\circ + W^\circ$ is the smallest subspace of V' containing U° and W° to show that $(U \cap W)^\circ$ contains $U^\circ + W^\circ$ it suffices to show that $U^\circ \subseteq (U \cap W)^\circ$ and $W^\circ \subseteq (U \cap W)^\circ$. But, both of these inclusions are obvious since if φ annihilates all of U and/or W it annihilates all of $U \cap W$.

Conversely, we want to show that $(U \cap W)^\circ \subseteq U^\circ + W^\circ$. Here's one way to do this. Let $t_1, \ldots, t_k$ be a basis of $U \cap W$. Extend this to a basis $\{t_1, \ldots, t_k, u_1, \ldots, u_\ell\}$ of U and $\{t_1, \ldots, t_k, w_1, \ldots, w_m\}$ of W. I claim that the the list $\{t_1, \ldots, t_k, u_1, \ldots, u_\ell, w_1, \ldots, w_m\}$ consists of independent vectors. To see this, suppose that

$$\sum_{a=1}^{k} \alpha_a t_a + \sum_{b=1}^{\ell} \beta_b w_b + \sum_{c=1}^{m} \gamma_c w_c = 0 \tag{2}$$

Note then that we get the equality

$$\sum_{a=1}^{k} \alpha_a t_a + \sum_{b=1}^{\ell} \beta_b w_b = \sum_{c=1}^{m} -\gamma_c w_c \tag{3}$$

Since the left hand side is evidently in U and the right hand side is evidently in W the equality implies that both (equal) sides are in $U \cap W$. In particular, since the left hand side is in $U \cap W$ we see by construction that the $\beta_b = 0$ for all b. But, we then deduce that α_a and γ_c are all zero by the independence of $\{t_1, \ldots, t_k, w_1, \ldots, w_m\}$. Finally, since this list is independent we can extend it to a basis $\{t_1, \ldots, t_k, u_1, \ldots, u_\ell, w_1, \ldots, w_m, v_1, \ldots, v_n\}$ of V. Now any $\varphi \in V'$ can be expressed uniquely in the form

$$\varphi = \sum_{a=1}^{\ell} \alpha_a t'_a + \sum_{b=1}^{\ell} \beta_b u'_b + \sum_{c=1}^{m} \gamma_c w'_c + \sum_{d=1}^{n} \delta_d v'_d \tag{4}$$

It's clear then that if $\varphi \in (U \cap W)^\circ$ means precisely that $\alpha_a = 0$ for all a. Note then that we can write

$$\begin{aligned} \varphi &= \sum_{a=1}^{\ell} 0 t'_a + \sum_{b=1}^{\ell} \beta_b u'_b + \sum_{c=1}^{m} \gamma_c w'_c + \sum_{d=1}^{n} \delta_d v'_d \\ &= \underbrace{\sum_{b=1}^{\ell} \beta_b u'_b}_{\psi_1} + \underbrace{\sum_{c=1}^{m} \gamma_c w'_c + \sum_{d=1}^{n} \delta_d v_d}_{\psi_2} \end{aligned} \tag{5}$$

Now, note that since the coefficients of $t'_1, \ldots, t'_k, w'_1, \ldots, w'_m\}$ are zero in the expression for ψ_1 we see that $\psi_1 \in W^\circ$. Since the coefficients of $t'_1, \ldots, t'_k, u'_1, \ldots, u'_\ell$ are zero in the expression for ψ_2 we see that $\psi_2 \in W^\circ$. Since $\varphi = \psi_1 + \psi_2$, we're done.

Prob 2. Suppose that V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and $\text{null}(T') = \text{span}(\varphi)$ for some $\varphi \in W'$. Prove that $\text{range}(T) = \text{null}(\varphi)$. Give an example of such a pair with $T \neq 0$, $\varphi \neq 0$, and with $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

Solution: If $\varphi = 0$ this is trivial, so we assume that $\varphi \neq 0$.

Let us first show that $\text{range}(T) \subseteq \text{null}\varphi$. This means precisely that $\varphi \circ T = 0$. Indeed $\varphi \circ T = 0$ means that for all $v \in V$ we have htat $(\varphi \circ T)(v) = \varphi(T(v)) = 0$. But this just means that for all w of the form $T(v)$ we have that $\varphi(w) = 0$. But this just means that $\text{range}(T) \subseteq \text{null}(\varphi)$. Now, $\varphi \circ T = T'(\varphi)$. Since $\varphi \in \text{null}(T')$ we get $T'(\varphi) = \varphi \circ T = 0$ as desired.

To show equality, since all spaces involved are finite-dimensional, it suffices to show that $\dim \text{range}(T) = \dim \text{null}(\varphi)$. Since $\varphi \neq 0$ it's evidently a surjective linear map $W \to F$ so that, by the Rank-Nullity theorem, we have that $\dim \text{null}(\varphi) = \dim W - 1$. Thus, we're really trying to show that $\dim \text{range}(T) = \dim W - 1$. But, by assumption we have that $\dim \text{null}(T') = \text{span}(\varphi)$. Since $\varphi \neq 0$ we have that $\dim \text{span}(\varphi) = 1$ so that $\dim \text{null}(T') = 1$. But, we know that $\dim \text{null}(T') = \dim W - \dim \text{range}(T)$. So, we deduce that $\dim \text{range}(T) = \dim W - 1$ as desired.

To produce an example, consider the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(a, b) = (a, b, a + b)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ given by $\varphi(a, b, c) = a + b - c$. Note then that evidently $\text{null}(\varphi) = \text{range}(T)$.

Prob 3. Prove that $\mathcal{P}(\mathbb{R})'$ and $\mathbb{R}^\infty$ are isomorphic.
Solution: Note that $\{1, x, x^2, \ldots, \}$ is a basis of $\mathcal{P}(\mathbb{R})$. Then, by the obvious analogue of Theorem 3.5 of Axler (which goes through without comment) a linear map $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is just an assignement of scalars a_n to each x^n for $n = 0, 1, \ldots$. This is precisely $\mathbb{R}^\infty$. More explicitly, the isomorphism $T : \mathcal{P}(\mathbb{R})' \to \mathbb{R}^\infty$ is the map $T(\varphi) = (\varphi(1), \varphi(x), \varphi(x^2), \ldots)$.

Prob 4. Prove that every polynomial with real coefficients and odd degree has a (real) root.
Solution: Let $p(x)$ be such a polynomial. Since $p(x)$ has a root if and only if $-p(x)$ has a root we may assume without loss of generality that the leading coefficient of $p(x)$ is positive. Then, we evidently have that $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. In particular, for x sufficiently large $p(x) > 0$ and for x sufficiently negative $p(x) < 0$. Let's, in particular, pic $x_0 < 0$ such that $p(x_0) < 0$ and $x_1 > 0$ such that $p(x_1) > 0$. Then, since $0 \in [p(x_0), p(x_1)]$ and $p(x)$ is continuous we know from the Intermediate Value Theorem that $p(x)$ takes the value 0 somewhere on $[x_0, x_1]$. The conclusion follows.

Prob 5.+Prob. 6 I'm just going to combine these into proving the following statement. Let us fix distinct elements $x_0, \ldots, x_n$ of $\mathbb{C}$. Then, for any $y_0, \ldots, y_n \in \mathbb{C}$ there exists a unique polynomial $p(x) \in \mathcal{P}_n(\mathbb{C})$ such that $p(x_i) = y_i$ for $i = 0, \ldots, n$. If $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ are real, then $p(x) \in \mathcal{P}_n(\mathbb{R})$.
Solution: Let us set $\mathbf{x} := (x_0, \ldots, x_n)$ and define the $(n+1) \times (n+1)$ matrix $A_{\mathbf{x}}$ to have entries $(a_{ij}) = (x_i^j)$. Now, if $v = (a_0, \ldots, a_n) \in V := \mathbb{C}^{n+1}$ note then that we have the following equality

$$A_{\mathbf{x}} v = (p_v(x_0), \ldots, p_v(x_n)) \tag{6}$$

where p_v is the polynomial with coefficient vector v (i.e. $p_v(x) = a_0 + a_1 x + \cdots + a_n x^n$). Note then that the map $v \mapsto p_v(x)$ is a (linear) bijection $V \to \mathcal{P}_n(\mathbb{C})$. It's then easy to see that the claim that there exist a unique polynomial $p(x) \in \mathcal{P}_n(\mathbb{C})$ such that $p(x_i) = y_i$ for all i is equivalent to the claim that there is a unique $v \in V$ such that $A_{\mathbf{x}} v = (y_0, \ldots, y_n)$.

To prove this it suffices to show that $A_{\mathbf{x}}$ is invertible. Since $A_{\mathbf{x}}$ is square, it suffices to show that $A_{\mathbf{x}}$ is injective. But, if $A_{\mathbf{x}} v = A_{\mathbf{x}} w$ then this means that $p_v(x_i) = p_w(x_i)$ for $i = 0, \ldots, n$. Note then that if we set $q(x) := p_v(x) - p_w(x)$ then $q(x_i) = 0$ for $i = 0, \ldots, n$. By Corollary 4.3 of Axler this implies that $q(x) = 0$ and thus that $p_v(x) = q_v(x)$. This in turn implies that $v = w$ as desired. In particular, we see that the polynomial $p(x)$ such that $p(x_i) = y_i$ is $p_v(x)$ with $v := A_{\mathbf{x}}^{-1}(y_0, \ldots, y_n)$.

Now, if $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ are real then we claim that the polynomial $p(x)$ such that $p(x_i) = y_i$ has real coefficients. By the last sentence of the last paragraph we know that the coefficient vector for this $p(x)$ is $A_{\mathbf{x}}^{-1}(y_0, \ldots, y_n)$. Since $x_0, \ldots, x_n$ are real, the matrix $A_{\mathbf{x}}$ consists of real entries, and thus so does $A_{\mathbf{x}}^{-1}$. Since $(y_0, \ldots, y_n)$ is a vector of real numbers we deduce that the coefficient vector $A_{\mathbf{x}}^{-1}(y_0, \ldots, y_n)$ is a vector of real numbers as desired.