Math 110, Spring 2019.

Homework 6 solutions.

Prob 1. Let V be a vector space over $\mathbb{F}$. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic.

Proof. Define $\mathcal{I} : \mathcal{L}(\mathbb{F}, V) \to V$ by the formula $T \mapsto T(1)$, i.e., $\mathcal{I}(T) := T(1)$. Then, for all $\lambda \in \mathbb{F}$ and all $T_1, T_2 \in \mathcal{I} : \mathcal{L}(\mathbb{F}, V)$, we have

$$\mathcal{I}(T_1 + \lambda T_2) = (T_1 + \lambda T_2)(1) = T_1(1) + \lambda T_2(1) = \mathcal{I}(T_1) + \lambda \mathcal{I}(T_2).$$

So $\mathcal{I}$ is a linear map from $\mathcal{L}(\mathbb{F}, V)$ to V. Define $\mathcal{J} : V \to \mathcal{L}(\mathbb{F}, V)$ by the formula $(\mathcal{J}v)(x) := xv$ for an arbitrary $x \in \mathbb{F}$. The map $\mathcal{J}$ is also linear because, for any $v_1, v_2 \in V$ and any $\lambda, x \in \mathbb{F}$, we have

$$\mathcal{J}(v_1 + \lambda v_2)(x) = (v_1 + \lambda v_2)x = xv_1 + x\lambda v_2 = \mathcal{J}(v_1)(x) + \lambda \mathcal{J}(v_2)(x).$$

Now observe that

$$\mathcal{I}\mathcal{J}(v) = (\mathcal{J}v)(1) = 1 \cdot v = v \qquad \text{for all } v \in V,$$

and that

$$(\mathcal{J}\mathcal{I}(T))(x) = (\mathcal{J}(T(1)))(x) = T(1)x = T(x) \quad \text{for all } T \in \mathcal{L}(\mathbb{F}, V) \text{ and all } x \in \mathbb{F}.$$

So $\mathcal{J}$ is the inverse of $\mathcal{I}$, so $\mathcal{I}$ (and $\mathcal{J}$) is an isomorphism between V and $\mathcal{L}(\mathbb{F}, V)$.

Prob 2. Give an example of V, W, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$ such that

(a) $TS = I$ but S is not invertible.

(b) $TS = I$ but T is not invertible.

Example for both (a) and (b). Let $V = W = \mathcal{P}(\mathbb{R})$, let S be the differentiation operator $p(x) \mapsto p'(x)$, and let T be the integration operator $p(x) \mapsto \int_0^x p(t)dt$. We know from Calculus that both are linear operators on V and that $TS = I$, i.e.,

$$\frac{d}{dx}\int_0^x p(t)dt = p(x) \qquad \text{(Fundamental Theorem of Calculus).}$$

However, neither T nor S is invertible on V. Indeed, $S(1) = 0$, so S is not injective, hence not invertible. Also, any polynomial in the range of T takes the value zero at the origin because $\int_0^0 p(t)dt = 0$ for any $p \in V$. So range $T \not\ni 1$. So, T is not surjective, hence not invertible.

Remark: many other examples are possible.

Prob 3. Let $V = \mathcal{P}_2(\mathbb{R})$ and suppose $\varphi_j(p) = p(j)$, $j = 0, 1, 2$. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is a basis for V' and find a basis (p_0, p_1, p_2) of $\mathcal{P}_2(\mathbb{R})$ whose dual is $(\varphi_0, \varphi_1, \varphi_2)$.

Solution. Let $p_0(x) := (x-1)(x-2)/2$, $p_1(x) := -x(x-2)$, $p_2 := x(x-1)/2$. These polynomials have degree 2, so they belong to V. Moreover, $p_0(0) = 1$, $p_0(1) = p_0(2) = 0$, $p_1(0) = p_1(2) = 0$, $p_1(1)1$, $p_2(0) = p_2(1) = 0$, $p_2(2) = 1$. In other words $\varphi_j(p_i) = \delta_{ij}$ for i, $j = 1, 2, 3$, i.e., the duality conditions hold.

Suppose there exist scalars λ_0, λ_1, $\lambda_2 \in \mathbb{R}$ such that $\lambda_0\varphi_0 + \lambda_1\varphi_1 + \lambda_2\varphi_2 = 0$. Then

$$\begin{aligned} 0 &= (\lambda_0\varphi_0 + \lambda_1\varphi_1 + \lambda_2\varphi_2)(p_0) = \lambda_0 \\ 0 &= (\lambda_0\varphi_0 + \lambda_1\varphi_1 + \lambda_2\varphi_2)(p_1) = \lambda_1 \\ 0 &= (\lambda_0\varphi_0 + \lambda_1\varphi_1 + \lambda_2\varphi_2)(p_2) = \lambda_2. \end{aligned}$$

So $\lambda_j = 0$ for all $j = 1, 2, 3$, hence the functionals φ_0, φ_1, φ_2 are linearly independent. Since there are three of them and since $\dim V = \dim V' = 3$ by 3.95, these functionals form a basis for V' by 2.39.

Now suppose there exist scalars α_0, α_1, $\alpha_2 \in \mathbb{R}$ such that $\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 = 0$. Then

$$\begin{aligned} 0 &= \varphi_0(\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2) = \alpha_0 \\ 0 &= \varphi_1(\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2) = \alpha_1 \\ 0 &= \varphi_2(\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2) = \alpha_2. \end{aligned}$$

So, $\alpha_j = 0$ for all $j = 1, 2, 3$, so the polynomials p_0, p_1, p_2 are linearly independent. Since there are three of them and since $\dim V = 3$, these polynomials form a basis for V by 2.39.

This establishes all claims we needed to verify.

Prob 4. Let V be a finite-dimensional vector space and let U be its proper subspace (i.e., $U \neq V$). Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$.

Solution. Let $v_1, \ldots, v_k$ be an arbitrary basis of U and let $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ be its arbitrary extension to a basis of V (which is guaranteed by 2.33). Since U is a proper subspace of V, $n > k$.

Define $\varphi \in V'$ via its values on the basis $v_1, \ldots, v_n$:

$$\varphi(v_j) = \delta_{jn}.$$

That is, the values of φ on all basis vectors v_j are required to be zero except on the vector v_n. By 3.5, this defines a linear map from V to $\mathbb{F}$, i.e., a linear functional. Then

$$\varphi(\sum_{j=1}^{k} \alpha_j v_j) = \sum_{j=1}^{k} \alpha_j \varphi(v_j) = 0.$$

Since $U = \operatorname{span}(v_1, \ldots, v_k)$, this shows that $\varphi(u) = 0$ for all $u \in U$. On the other hand, $\varphi(v_n) = 1 \neq 0$, so φ is not the zero functional. This concludes the proof.

Prob 5. Let $T : \mathcal{P}(\mathrm{I\!R}) \to \mathcal{P}(\mathrm{I\!R}) : p(x) \mapsto (x-1)^3 p(x) + p''(x)$.

(a) Let $\varphi \in \mathcal{P}(\mathrm{I\!R})' : \varphi(p) = p'(1)$. Give a formula for $T'(\varphi)$.

(b) Let $\varphi \in \mathcal{P}(\mathrm{I\!R})' : \varphi(p) = \int_0^1 p(x)\,dx$. Evaluate $T'(\varphi)(x^2)$.

Solution. (a) We have $T'(\varphi)(p) = (\varphi \circ T)(p) = \varphi(Tp)$, hence

$$T'(\varphi)(p) = ((x-1)^3 p(x) + p''(x))'|_{x=1} = (3(x-1)^2 p(x) + (x-1)^3 p'(x) + p'''(x))|_{x=1} = p'''(1).$$

(b) The same general formula $T'(\varphi)(p) = (\varphi \circ T)(p) = \varphi(Tp)$ holds, so

$$T'(\varphi)(x^2) = \int_0^1 ((t-1)^3 t^2 + 2)dt = \int_0^1 (t-1)^3((t-1)^2 + 2(t-1) + 1)dt + 2 = -\frac{1}{6} + \frac{2}{5} - \frac{1}{4} + 2 = \frac{119}{60}.$$