Math 110, Fall 2015.

Homework 5, due March 2.

Prob 1. Let $V := \mathbb{C}^3$. Give an example of a map $T \in \mathcal{L}(V, V)$ such that $V = \operatorname{null} T \oplus \operatorname{range} T$ or prove that none exists.

Solution1.

All the followings work (the first two are trivial examples as both $\operatorname{null} T$ and $\operatorname{range} T$ are not proper):

(1) $T = 0$, then $\operatorname{null} T = V$ and $\operatorname{range} T = \{(0,0,0)\}$.

(2) $T = \mathrm{id}_V$, then $\operatorname{null} T = \{(0,0,0)\}$ and $\operatorname{range} T = V$.

(3) $T((x,y,z)) = (0,y,z)$, then

$$\begin{aligned}\operatorname{null} T &= \{(x,y,z) : T((x,y,z)) = (0,0,0)\} = \{(x,y,z) : (0,y,z) = (0,0,0)\}\\ &= \{(x,0,0)\} = \operatorname{span}((1,0,0))\\ \operatorname{range} T &= \{T((x,y,z))\} = \{(0,y,z)\} = \{y(0,1,0) + z(0,0,1)\}\\ &= \operatorname{span}((0,1,0),(0,0,1))\end{aligned}$$

and $\operatorname{null} T + \operatorname{range} T = \{(x,0,0) + (0,y,z)\} = \{(x,y,z)\} = V$, $\operatorname{null} T \cap \operatorname{range} T = \{(x,y,z) : y = z = 0, x = 0\} = \{(0,0,0)\}$ implies $V = \operatorname{null} T \oplus \operatorname{range} T$.

Solution2. (Optional)

This solution tries to show that every example of T must have a matrix representation in some special form. And every T which has a matrix representation in one of the special forms is an example.

<u>Claim1</u>: Given $T \in \mathcal{L}(V,V)$ and $\dim V = n$. If $U, W \subseteq V$ invariant under T so that $U \oplus W = V$, let's consider bases of U and W, which are $(u_1, \ldots, u_m)$ and $(w_1, \ldots, w_{n-m})$, respectively, then $(u_1, \ldots, u_m, w_1, \ldots, w_{n-m})$ is a basis of V and $\mathcal{M}(T) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where $A = \mathcal{M}(T|_U)$ with respect to $(u_1, \ldots, u_m)$ and $B = \mathcal{M}(T|_W)$ with respect to $(w_1, \ldots, w_{n-m})$. Conversely, if $\exists\,(v_1, \ldots, v_n)$ basis of V such that $\mathcal{M}(T) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $A \in \mathbb{F}^{m,m}$, $B \in \mathbb{F}^{n-m,n-m}$. Then $U = \operatorname{span}(v_1, \ldots, v_m)$ and $W = \operatorname{span}(v_{m+1}, \ldots, v_n)$ are invariant under T so that $A = \mathcal{M}(T|_U)$, $B = \mathcal{M}(T|_W)$ and $U \oplus W = V$.

Proof. All the matrix representations are easy to verify by definition. We just need to verify (1) bases can be chosen as described (2) for the converse part, $U \oplus W = V$.

(1) $\dim V = \dim(U \oplus W) = \dim U + \dim W - \dim U \cap W = \dim U + \dim W$, hence we can really have bases of U and W with correct *amount*. And $V = U \oplus W = \{\sum a_i u_i + \sum b_j w_j\} = \operatorname{span}(u_1, \ldots, u_m, w_1, \ldots, w_{n-m})$. By 2.42, $(u_1, \ldots, u_m, w_1, \ldots, w_{n-m})$ must be a basis of V.

(2) $U + W = \left\{ \sum\limits_{i \le m} a_i v_i + \sum\limits_{j > m} b_j v_j \right\} = V$ and $U \cap W = \left\{ v \in V : v = \sum\limits_{i \le m} a_i v_i = \sum\limits_{j > m} b_j v_j \right\} = \{0\}$ because (v_i) is linearly independent, $a_i = b_j = 0$. Hence $V = U \oplus W$.

<u>Claim2</u>: Given $T \in \mathcal{L}(V,V)$ and $\dim E(\lambda, T) = m \neq 0$. Then $\mathcal{M}(T|_{E(\lambda,T)}) = \lambda I_m$ for every basis of $E(\lambda, T)$.

Proof. For any basis $(v_1, \ldots, v_m)$ of $E(\lambda, T)$, $Tv_i = \lambda v_i$, hence $\mathcal{M}(T|_{E(\lambda,T)}) = \lambda I_m$ by definition.

If such T exists, since both $\operatorname{null} T$, $\operatorname{range} T$ are invariant under T and we need them proper, Claim1 says such T satisfies $\mathcal{M}(T) = \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ or $\mathcal{M}(T) = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$ for some basis constructed from bases of $\operatorname{null} T$ and $\operatorname{range} T$. Moreover, $\operatorname{null} T = E(0, T)$, by Claim2, we need the upper-left block to be the zero matrix. Furthermore, from the form of $\mathcal{M}(T)$, one can see $\dim \operatorname{range}(T|_{\operatorname{range} T}) = \dim \operatorname{range} T$, hence $T|_{\operatorname{range} T}$ must be surjective, and hence invertible by 3.69. To sum up, if such T exists, then

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$$

with the lower block matrix invertible.
If T has a matrix representation in the first case with respect to (v_1, v_2, v_3), then

$$\begin{aligned}
\operatorname{null} T &= \{v \in V : Tv = 0\} = \{v \in V : \mathcal{M}(T)\mathcal{M}(v) = \mathcal{M}(0)\} \\
&= \left\{ av_1 + bv_2 + cv_3 : \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \{av_1\} = \operatorname{span}(v_1) \\
\operatorname{range} T &= \{T(av_1 + bv_2 + cv_3)\} = \{\mathcal{M}^{-1}(\mathcal{M}(T(av_1 + bv_2 + cv_3)))\} \\
&= \left\{ \mathcal{M}^{-1}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \right\} = \left\{ \mathcal{M}^{-1}\left(\begin{bmatrix} 0 \\ b' \\ c' \end{bmatrix} \right) : b', c' \in \mathbb{F} \right\} \\
&= \{b'v_2 + c'v_3\} = \operatorname{span}(v_2, v_3)
\end{aligned}$$

Hence $\operatorname{null} T + \operatorname{range} T = \{av_1 + (bv_2 + cv_3)\} = V$ and $\operatorname{null} T \cap \operatorname{range} T = \{v \in V : v = av_1 = bv_2 + cv_3\} = \{0\}$ because (v_i) is linearly independent, implying $V = \operatorname{null} T \oplus \operatorname{range} T$. Similar argument works for T having a matrix representation in the second case. In particular, consider $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$ and $\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow Tv_1 = (0, 0, 0), Tv_2 = v_2, Tv_3 = v_3$, then we recover (3) in **Solution1**.

Prob 2. Given an example of a map $T \in \mathcal{L}(\mathbb{R}^6, \mathbb{R}^2)$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 = x_2, \ \ x_3 + x_6 = 0, \ \ x_1 + x_3 - x_5 = 0\}$$

or prove that none such exists.
Solution.

$$\begin{aligned} \operatorname{null} T &= \{(x_1, x_1, x_3, x_4, x_1 + x_3, -x_3)\} \\ &= \{x_1(1, 1, 0, 0, 1, 0) + x_3(0, 0, 1, 0, 1, -1) + x_4(0, 0, 0, 1, 0, 0)\} \\ &= \operatorname{span}((1, 1, 0, 0, 1, 0), (0, 0, 1, 0, 1, -1), (0, 0, 0, 1, 0, 0)) \end{aligned}$$

If such T exists, then by 3.22 (rank-nullity), $6 = \dim \mathbb{R}^6 = \dim \operatorname{null} T + \dim \operatorname{range} T \leq 3 + \dim \mathbb{R}^2 = 5$, contradiction. Hence none such exists.

Prob 3. Let $T : f \mapsto f'' - 2f' + f$. Write down its matrix representation

(a) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using the standard monomial basis both for the domain and codomain;

(b) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using the standard monomial basis both for the domain and codomain;

(c) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using a Newton basis $1, x, x(x-1)$ for the domain and the standard monomial basis for the codomain;

(d) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using a shifted monomial basis $1, x-1, (x-1)^2, (x-1)^3$ for the domain and a Newton basis $1, x-1, (x-1)x, (x-1)x(x+1)$ for the codomain.

Solution.

(a)

$$\begin{aligned} T(1) &= 1 && = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ T(x) &= -2 + x && = (-2)\cdot 1 + 1\cdot x + 0\cdot x^2 \\ T(x^2) &= 2 - 4x + x^2 && = 2\cdot 1 + (-4)\cdot x + 1\cdot x^2 \end{aligned} \Rightarrow \mathcal{M}(T) = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{aligned} T(1) &= 1 && = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x) &= -2 + x && = (-2)\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x^2) &= 2 - 4x + x^2 && = 2\cdot 1 + (-4)\cdot x + 1\cdot x^2 + 0\cdot x^3 \\ T(x^3) &= 6x - 6x^2 + x^3 && = 0\cdot 1 + 6\cdot x + (-6)\cdot x^2 + 1\cdot x^3 \end{aligned} \Rightarrow \mathcal{M}(T) = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{aligned} T(1) &= 1 && = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ T(x) &= -2 + x && = (-2)\cdot 1 + 1\cdot x + 0\cdot x^2 \\ T(x(x-1)) &= 2 - 2(2x-1) + x(x-1) && = 4\cdot 1 + (-5)\cdot x + 1\cdot x^2 \end{aligned} \Rightarrow \mathcal{M}(T) = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\begin{aligned} T(1) &= 1 = 1\cdot 1 + 0\cdot (x-1) + 0\cdot (x-1)x + 0\cdot (x-1)x(x+1) \\ T(x-1) &= -2 + (x-1) = (-2)\cdot 1 + 1\cdot (x-1) + 0\cdot (x-1)x + 0\cdot (x-1)x(x+1) \\ T((x-1)^2) &= 2 - 2(2(x-1)) + (x-1)^2 \\ &= 2 - 4(x-1) + (x-1)x - (x-1) \\ &= 2\cdot 1 + (-5)\cdot (x-1) + 1\cdot (x-1)x + 0\cdot (x-1)x(x+1) \\ T((x-1)^3) &= 6(x-1) - 2(3(x-1)^2) + (x-1)^3 \\ &= 6(x-1) - 6[(x-1)x - (x-1)] + [(x-1)x(x+1) - 3(x-1)x + (x-1)] \\ &= 0\cdot 1 + 13\cdot (x-1) + (-9)\cdot (x-1)x + 1\cdot (x-1)x(x+1) \end{aligned}$$

$$\Rightarrow \mathcal{M}(T) = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prob 4. Suppose V and W are finite-dimensional vector spaces. Let $v \in V$, and consider

$$E := \{T \in \mathcal{L}(V,W) : Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V,W)$.

(b) Suppose $v \neq 0$. What is $\dim E$?

Solution1.

(a) $0 \in E$ and $\forall\, T_1, T_2 \in E$, $c \in \mathbb{F}$, $(cT_1 + T_2)(v) = cT_1(v) + T_2(v) = 0 \Rightarrow cT_1 + T_2 \in E$. Hence E is a subspace of $\mathcal{L}(V,W)$. (closed under addition by $c = 1$, closed under scalar multiplication by $T_2 = 0$)

(b) Consider $S : \mathcal{L}(V,W) \to W$ defined by $T \mapsto Tv$. Then S is linear because $S(T_1 + cT_2) = (T_1 + cT_2)(v) = T_1 v + cT_2 v = S(T_1) + cS(T_2)$, for any $T_1, T_2 \in \mathcal{L}(V,W)$ and $c \in \mathbb{F}$. Moreover by definition, $\operatorname{null} S = E$. And S is surjective because $\forall\, w \in W$, by 3.5, one can construct $T_w \in \mathcal{L}(V,W)$ defined by

$$\begin{aligned} v &\mapsto w \\ v_2 &\mapsto 0 \\ \vdots\ & \quad \vdots \\ v_n &\mapsto 0 \end{aligned}$$

where $(v, v_2, \cdots, v_n)$ is a basis of V obtained by extending the linearly independent list (v), by 2.33. Therefore $S(T_w) = T_w(v) = w$. In conclusion, by 3.61, we have

$$\begin{aligned} \dim E = \dim \operatorname{null} S &= \dim \mathcal{L}(V,W) - \dim \operatorname{range} S \\ &= (\dim V)(\dim W) - \dim W \end{aligned}$$

Solution2.

(a) The same as in **Solution1**.

(b) Consider $(v, v_2, \cdots, v_n)$ basis of V obtained by extending the linearly independent list (v), by 2.33. Suppose W has a basis $(w_1, \cdots, w_m)$. Then we can describe E in a clearer way by looking at matrix representations using these two bases:

$$\begin{aligned} E &= \{T \in \mathcal{L}(V,W) : Tv = 0\} = \mathcal{M}^{-1}\left\{\mathcal{M}(T) \in \mathbb{F}^{m,n} : \mathcal{M}(T)\mathcal{M}(v) = \mathcal{M}(0)\right\} \\ &= \mathcal{M}^{-1}\left\{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix} : \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}1 \\ 0 \\ \vdots \\ 0\end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0\end{bmatrix}\right\} \\ &= \mathcal{M}^{-1}\left\{\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & \cdots & a_{mn}\end{bmatrix}\right\} \\ &= \mathcal{M}^{-1}\left\{\sum_{1\le i\le m, 2\le j\le n} a_{ij}E_{ij}\right\} = \left\{\sum_{1\le i\le m, 2\le j\le n} a_{ij}\mathcal{M}^{-1}(E_{ij})\right\} = \operatorname{span}(\mathcal{M}^{-1}(E_{ij}))_{1\le i\le m, 2\le j\le n} \end{aligned}$$

where $E_{ij} \in \mathbb{F}^{m,n}$ whose entries are all 0 except for a 1 at the entry which is in the i-th row and the j-th column. It is easy to verify E_{ij} are linearly independent, then since $\mathcal{M}^{-1}$ is an isomorphism, $\mathcal{M}^{-1}(E_{ij})$ are also linearly independent. ($\sum b_{ij}\mathcal{M}^{-1}(E_{ij}) = 0 \Rightarrow \mathcal{M}^{-1}\left(\sum b_{ij}E_{ij}\right) = 0 \Rightarrow \sum b_{ij}E_{ij} = 0 \Rightarrow b_{ij} = 0$) Hence $\dim E = m(n-1) = mn - m$.

Prob 5. Call a matrix representation *optimally sparse* if it consists of zeros and ones only and the number of ones is as small as possible. Let $V = \mathcal{P}_4(\mathbb{R})$ and let $T \in \mathcal{L}(V, V) : f \mapsto f(x+2) - 2f(x+1) + f(x)$.

Scenario 1: Given the freedom to use any two bases for V as a domain and a co-domain, does T have an optimally sparse representation? If so, find it.

Scenario 2: Given the freedom to use any *single* basis for V *both* as a domain and a co-domain, does T have an optimally sparse representation? If so, find it.

Solution.
Claim: (Exercise 3.C.1) Suppose $T \in \mathcal{L}(V, W)$ and $\dim V = n$. If $\dim \operatorname{null} T = m$, then for any matrix representation, there are at least $n - m = \dim \operatorname{range} T$ nonzero entries.
Proof. If not, say there is a matrix representation $\mathcal{M}(T)$ using basis $(v_1, \cdots, v_n)$ of V with k nonzero entries, where $k < n - m$. Then there are at most k columns in $\mathcal{M}(T)$ which are not zero vectors, i.e., at least $n - k$ columns in $\mathcal{M}(T)$ which are all zero vectors. Say the $a_1, \ldots, a_{n-k}$-th columns are zero vectors. Then $\operatorname{null} T \supset \operatorname{span}(v_{a_1}, \cdots, v_{a_{n-k}})$ and $\dim \operatorname{null} T \geq n - k > m$, contradiction.

By this claim, since

$$\begin{aligned}
\operatorname{null} T &= \{f \in \mathcal{P}_4(\mathbb{R}) : Tf = 0\} \\
&= \{a + bx + cx^2 + dx^3 + ex^4 : aT(1) + bT(x) + cT(x^2) + dT(x^3) + eT(x^4) = 0\} \\
&= \{a + bx + cx^2 + dx^3 + ex^4 : c \cdot 2 + d(6x + 6) + e(12x^2 + 24x + 14) = 0\} \\
&= \{a + bx + cx^2 + dx^3 + ex^4 : 12ex^2 + (24e + 6d)x + (14e + 6d + 2c) = 0\} \\
&= \{a + bx\} = \operatorname{span}(1, x)
\end{aligned}$$

is 2-dimensional, so an optimally sparse matrix representation, if exists, must have at least 3 nonzero entries.

(1) If we start from the standard basis std $= (1, x, x^2, x^3, x^4)$ of $\mathcal{P}_4(\mathbb{R})$, we have $T(1) = T(x) = 0$ and $T(x^2) = 2$, $T(x^3) = 6x+6$, $T(x^4) = 12x^2+24x+14$. One can see that $(T(x^2), T(x^3), T(x^4))$ is linearly independent because they have different degrees. We can extend it to $B = (T(x^2), T(x^3), T(x^4), x^3, x^4)$, which is linearly independent because again they have different degrees. It is a basis by 2.39. Then the matrix representation of T using std in the domain and B in the codomain is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is optimally sparse because we cannot do better than this, as shown above.

(2) Suppose there is a basis $\tilde{B} = (v_1, \cdots, v_5)$ such that the matrix representation of T using $\tilde{B}$ in both domain and codomain is the same as in **(1)**, then $T(v_1) = T(v_2) = 0$, $v_1 = T(v_3)$, $v_2 = T(v_4)$, $v_3 = T(v_5)$, hence $\tilde{B} = (T^2(v_5), T(v_4), T(v_5), v_4, v_5)$ satisfying $T^3(v_5) = 0$ and $T^2(v_4) = 0$. Notice that for $f(x) = a + bx + cx^2 + dx^3 + ex^4$, we have

$$\begin{aligned}
T(f) &= 12ex^2 + (24e + 6d)x + (14e + 6d + 2c) \\
T^2(f) &= 24e \\
T^3(f) &= 0
\end{aligned}$$

Hence one can choose v_4 to be any polynomial whose degree at most 3 and v_5 can be chosen arbitrary. But to make $\tilde{B}$ linearly independent, it is natural to choose $v_4 = x^3$ and $v_5 = x^4$. Then $\tilde{B} = (24, 6x + 6, 12x^2 + 24x + 14, x^3, x^4)$, which is linearly independent because they have different degrees and hence a basis by 2.39, makes the matrix representation the same as **(1)** and optimally sparse.