Math 110, Spring 2019.

Homework 4 solutions.

Prob 1. Give, with proof, an example of three linear independent maps from $\mathcal{L}(V, W)$ where $V = W = \mathbb{R}^2$ or prove that no such example exists.

Example: Here is a simple example. Consider

$$T_1 : (x_1, x_2) \mapsto (x_1, x_2), \quad T_2 : (x_1, x_2) \mapsto (x_1, 0), \quad T_3 : (x_1, x_2) \mapsto (0, x_1).$$

These are all linear maps (the first being the identity) according to the discussion on the bottom of p. 53 of our textbook because each component of their output has the form $a_1x_1 + a_2x_2$ for some real constants a_1 and a_2. Now suppose that, for some α_1, α_2, $\alpha_3 \in \mathbb{R}$,

$$T := \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 = 0.$$

Let T act on $e_2 = (0, 1)$ to obtain

$$(0, 0) = Te_2 = \alpha_1(1, 0) = (\alpha_1, 0).$$

This implies $\alpha_1 = 0$. Now let T act on $e_1 = (1, 0)$. This gives

$$(0, 0) = Te_1 = \alpha_2(1, 0) + \alpha_3(0, 1) = (\alpha_2, \alpha_3).$$

This implies $\alpha_2 = \alpha_3 = 0$. Hence all α_j must be equal to zero. This shows T_1, T_2, T_3 are linearly independent.

Prob 2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Proof: Let $v \in V$ be arbitrary. Then $w := S(Tv) \in \operatorname{range} S$. Since $\operatorname{range} S \subset \operatorname{null} T$, it follows that $Tw = 0$. Hence

$$(ST)^2 v = STSTv = STw = S(0) = 0.$$

Thus all vectors $v \in V$ get mapped to zero by the transformation $STST$, Hence $STST$ is the zero map.

Prob 3. Suppose $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is defined by the formula $Tf = f'' + 3f'$. Check that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ and find a basis for the null space and a basis for the range of T.

Solution: The first differentiation decreases the degree of a polynomial by 1 and the second differentiation by 2, so indeed all polynomials of degree at most 3 get mapped to polynomials of degree at most 2 by T. T is linear because

$$T(f + \lambda g) = (f + \lambda g)'' + 3(f + \lambda g)' = f'' + \lambda g'' + 3f' + 3\lambda g' = (f'' + 3f') + \lambda(g'' + 3g') = Tf + \lambda Tg$$

for any f, $g \in \mathcal{P}_3(\mathbb{R})$ and any $\lambda \in \mathbb{R}$. Now let $p(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Then

$$(Tp)(x) = 2a_2 + 6a_3 x + 3(a_1 + 2a_2 x + 3a_3 x^2) = (2a_2 + 3a_1) + (6a_3 + 6a_2)x + 9a_3 x^2. \tag{1}$$

If $p(x) \in \operatorname{null} T$, then each coefficient of the resulting polynomial must equal zero for the polynomial to be identically zero. That implies $a_3 = 0$, hence $a_2 = 0$, hence $a_1 = 0$. The constant term a_0 is free. This shows that all constant functions, and only those, lie in the null space of T. Therefore, the constant function 1 forms a basis for $\operatorname{null} T$, and $\dim \operatorname{null} T = 1$.

Now (1) shows that $T(x/3) = 1$, $T(x^2/6 - x/9) = x$, $T(x^3/9 - x^2/9 + 2x/27) = x^2$, hence $\mathcal{P}_2(\mathbb{R}) \subseteq \operatorname{range} T$. But our co-domain is also $\mathcal{P}_2(\mathbb{R})$, hence

$$\mathcal{P}_2(\mathbb{R}) \subseteq \operatorname{range} T \subseteq \mathcal{P}_2(\mathbb{R}).$$

So $\operatorname{range} T = \mathcal{P}_2(\mathbb{R})$ and the list 1, x, x^2 (or any other basis for $\mathcal{P}_2(\mathbb{R})$) is also a basis for $\operatorname{range} T$.

Remark: Alternatively, we could have obtained a basis for $\operatorname{range} T$ by removing linearly dependent vectors from the list $T(1)$, $T(x)$, $T(x^2)$, $T(x^3)$, which follows from the proof of 3.22. In our particular case, $T(1) = 0$ and the remaining three polynomials are linearly independent [**as you can check**].

Prob 4. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Proof: Suppose there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W. Then $T(Sw) = TSw = w$ for all $w \in W$, hence each $w \in W$ has a pre-image Sw that is mapped by T to w. So T is surjective.

Now let us prove the converse. Let T be surjective, i.e., let $\operatorname{range} T = W$. Let $v_1, \ldots, v_n$ be a basis of V (recall that V is finite-dimensional by the assumption of our problem). By the proof of the Rank-Nullity Theorem 3.22, we know that the list $Tv_1, \ldots, Tv_n$ spans $\operatorname{range} T = W$. By 2.31, this list can be reduced to a basis of W. By renumbering the vectors $v_1, \ldots, v_n$ if necessary, we can assume that $Tv_1, \ldots, Tv_k$ is a basis of W for some $k \leq n$.

Define $S(Tv_j) = v_j$ for $j = 1, \ldots, k$ and extend S by linearity to the whole space W by 3.5. This defines a linear map from W to V. Now,

$$TS(Tv_j) = T(S(Tv_j)) = T(v_j) \qquad \text{for all } \ j = 1, \ldots, k.$$

In other words, the linear map $TS \in \mathcal{L}(W, W)$ acts as the identity map on all basis vectors $Tv_1, \ldots, Tv_k$ of W. Hence, by 3.5 again, the map TS is in fact the identity map on W.