Math 110, Spring 2019.

Homework 3 solutions.

Prob 1. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-2}^{2} p(x)\,dx = 0\}$.

(a) Find a basis for U.

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Solution. First let us prove the following

Lemma. Let $p_1(x), \ldots, p_k(x) \in \mathcal{P}_n(\mathbb{R})$ be polynomials of degrees $d_1 < d_2 < \cdots < d_k$ respectively. Then the list $(p_j(x))$ is linearly independent.

Proof. Suppose $\sum_{j=1}^{k} \alpha_j p_j(x) \equiv 0$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. If $\alpha_k \neq 0$, then the highest-degree term in $\sum_{j=1}^{k} \alpha_j p_j(x)$ is the same as the highest-degree term in $\alpha_k p_k(x)$, which has degree d_k. Hence α_k must equal zero. Now, if $\alpha_{k-1} \neq 0$, the highest-degree term in $\sum_{j=1}^{k} \alpha_j p_j(x)$ is the same as the highest-degree term in $\alpha_{k-1} p_{k-1}(x)$, which has degree d_{k-1}. Hence $\alpha_{k-1} = 0$. And so on. This shows that all α_j must in fact equal zero. Hence the initial list of polynomials is linearly independent. This proves the Lemma.

(a) Suppose a polynomial $p(x) = \sum_{j=0}^{4} a_j x^j$ belongs to U. Integrating $p(x)$ over $[-2, 2]$ and dividing the result by 4, we obtain

$$a_0 + \frac{4}{3} a_2 + \frac{16}{5} a_4 = 0.$$

Note that the polynomials x, $3x^2 - 4$, x^3, $5x^4 - 16$ all satisfy this condition. These polynomials have distinct degrees 1 through 4, so they are linearly independent by the Lemma above.

Note that $U \neq \mathcal{P}_4(\mathbb{R})$ since, e.g., the constant function 1 does not belong to U. Thus U is a proper subspace of $\mathcal{P}_4(\mathbb{R})$, hence its dimension cannot be equal to the dimension of $\mathcal{P}_4(\mathbb{R})$, which is equal to 5. Hence $\dim U \leq 4$. But the list x, $3x^2 - 4$, x^3, $5x^4 - 16$ of length four is linearly independent, so by 2.39, we conclude that this is a basis for U.

(b) If we add the constant 1 to this basis of U, we will obtain a list of five polynomials of all possible degrees from 0 to 4, which is therefore linearly independent by the Lemma. Since it contains $5 = \dim \mathcal{P}_4(\mathbb{R})$ vectors, we conclude by 2.39 that this list forms a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Take $W := \operatorname{span}\{1\}$. We already know that the list 1, x, $3x^2 - 4$, x^3, $5x^4 - 16$ is linearly independent, hence the zero polynomial can be expressed in only one way as a linear combination of these polynomials, hence the sum $W + U = W + \operatorname{span}(x, 3x^2 - 4, x^3, 5x^4 - 16) = \operatorname{span}(1, x, 3x^2 - 4, x^3, 5x^4 - 16)$ is direct.

Prob 2. Suppose $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 - w, v_2 - w, \ldots, v_m - w) \geq m - 1.$$

Proof. Let $U := \operatorname{span}(v_1 - w, v_2 - w, \ldots, v_m - w)$ and let $W := \operatorname{span}(w)$. Note that

$$U + W = \operatorname{span}(w, v_1 - w, v_2 - w, \ldots, v_m - w)$$

contains the vectors v_1 through v_m, which are linearly independent. Hence $\dim(U + W) \geq m$. Also, W is of dimension at most 1 because it is a span of a single vector.

Now, $m \leq \dim(U + W) = \dim U + \dim W - \dim(U \cap W)$, where the equality is due to 2.43. This implies

$$\dim U = \dim(U + W) + \dim(U \cap W) - \dim(W) \geq \dim(U + W) - \dim(W) \geq m - 1.$$

Prob 3. Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1+U_2+U_3) &= \dim(U_1)+\dim(U_2)+\dim(U_3)\\ &\quad -\dim(U_1\cap U_2)-\dim(U_1\cap U_3)-\dim(U_2\cap U_3)\\ &\quad +\dim(U_1\cap U_2\cap U_3)?\end{aligned}$$

Prove this formula or provide a counterexample.

Counterexample. Here is a fairly straightforward counterexample. Let $V = \mathbb{R}^2$, $u_1 := (1,0)$, $u_2 := (0,1)$, $u_3 := (1,1)$, and let $U_j := \operatorname{span}(u_j)$. Each U_j is 1-dimensional, being a span of a single non-zero vector. Next,

$$U_1+U_2+U_3 = \operatorname{span}(u_1,u_2,u_3) \supseteq \operatorname{span}(u_1,u_2) = V.$$

Note that none of the vectors u_j is in fact a scalar multiple of another, so any two of these vectors are linearly independent. Hence the zero vector can be represented only as a trivial linear combination of any pair of these vectors. By 1.44, this implies that

$$U_1\cap U_2 = U_1\cap U_3 = U_2\cap U_3 = \{0\}.$$

Finally, $U_1\cap U_2\cap U_3$ is a subspace of, say, $U_1\cap U_2$, which is the zero subspace. Hence

$$U_1\cap U_2\cap U_3 = \{0\}$$

as well. But then

$$\begin{aligned}\dim(U_1+U_2+U_3) = 2 \quad \neq \quad 3 &= \dim(U_1)+\dim(U_2)+\dim(U_3)-\dim(U_1\cap U_2)\\ &\quad -\dim(U_1\cap U_3)-\dim(U_2\cap U_3)+\dim(U_1\cap U_2\cap U_3).\end{aligned}$$

Prob 4. Let $a, b \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(4) + 5p'(2) + ap(1)p(3), \int_{-1}^{2} x^3 p(x)\, dx + b \cos p(0)).$$

Show that T is linear if and only if $a = b = 0$.

Proof. Suppose T is linear. Then we must have $T(0) = 0$ by 3.11. Plugging in $p \equiv 0$, we obtain $(0, b)$, which therefore must equl $(0, 0)$. This means $b = 0$. Now, by linearity of T again, we must have $T(2) = 2T(1)$. Plugging in $p \equiv 1$ and $p \equiv 2$, we obtain $T(2) = (4 + 4a, 2\int_{-1}^{2} x^3\, dx)$, $T(1) = (2 + a, \int_{-1}^{2} x^3\, dx)$. Now $T(2) = 2T(1)$ implies $4 + 4a = 2(2 + a)$, hence $a = 0$.

Conversely, suppose $a = b = 0$, hence $Tp = (2p(4) + 5p'(2), \int_{-1}^{2} x^3 p(x)\, dx)$ for all $p \in \mathcal{P}(\mathbb{R})$. Then, by the linearity of point evaluation, integration, and differentiation, we get

$$T(\lambda p) = (2\lambda q(4) + 5\lambda p'(2), \int_{-1}^{2} x^3 \lambda p(x)\, dx) = \lambda(2p(4) + 5p'(2), \int_{-1}^{2} x^3 p(x)\, dx) = \lambda T(p)$$

for any $\lambda \in \mathbb{R}$. By the linearity of point evaluation, integration and differentiation again,

$$\begin{aligned}
T(p+q) &= (2(p+q)(4) + 5(p+q)'(2), \int_{-1}^{2} x^3 (p+q)(x)\, dx) \\
&= (2(p(4) + q(4)) + 5(p'(2) + q'(2)), \int_{-1}^{2} x^3 (p(x) + q(x))\, dx) \\
&= (2p(4) + 2q(4) + 5p'(2) + 5q'(2), \int_{-1}^{2} x^3 p(x)\, dx + \int_{-1}^{2} x^3 q(x)\, dx) \\
&= (2p(4) + 5p'(2), \int_{-1}^{2} x^3 p(x)\, dx) + (2q(4) + 5q'(2), \int_{-1}^{2} x^3 q(x)\, dx) \\
&= T(p) + T(q).
\end{aligned}$$

So T is a linear map.

Prob 5. Suppose $T \in \mathcal{L}(V, W)$, $v_1, \ldots, v_m \in V$ and the list $Tv_1, Tv_2, \ldots Tv_m$ is linearly independent (in W). Prove that $v_1, \ldots, v_m$ must be linearly independent in V. What is the contrapositive of this statement?

Proof. Given that the list $Tv_1, Tv_2, \ldots Tv_m$ is linearly independent in W, suppose that a linear combination $\alpha_1 v_1 + \cdots + \alpha_m v_m$ is equal to zero (vector of V). By the linearity of T, we have

$$0 = T(0) = T(\alpha_1 v_1 + \cdots + \alpha_m v_m) = \alpha_1 T v_1 + \cdots + \alpha_m T v_m.$$

The linear independence of $Tv_1, Tv_2, \ldots, Tv_m$ now implies that $\alpha_1 = \cdots = \alpha_m = 0$. This shows that the list v_1, v_2, …, v_m is linearly independent in V.

The contrapositive of this statement is as follows: Suppose that the list $v_1, \ldots, v_m$ is linearly dependent in V. Then the list $Tv_1, Tv_2, \ldots, Tv_m$ is linearly dependent in W.

Prob 6. Suppose V is a nonzero finite-dimensional vector space and W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Proof. Let us prove the contrapositive, namely, that the finite dimensionality of $\mathcal{L}(V, W)$ implies the finite dimensionality of W. If $\dim \mathcal{L}(V, W) = N$, then any $N+1$ linear maps from V to W are linearly dependent.

Consider $N+1$ arbitrary vectors $w_j \in W$, $j = 1, \ldots, N+1$. Take any basis (v_k) of V. By 3.5, there exist linear maps $T_j \in \mathcal{L}(V, W)$ such that $T_j(v_1) = w_j$ for all $j = 1, \ldots, N+1$ (note that we do not care where the basis vectors other than v_1 are being sent). Since the list of maps T_1, …, T_{N+1} is linearly dependent in $\mathcal{L}(V, W)$, the list $T_1 v$, …, $T_{N+1} v$ is linearly dependent in W for any $v \in V$. Therefore, the list

$$w_1 (= T_1(v_1)),\ w_2 (= T_2(v_1)),\ \ldots,\ w_{N+1} (= T_{N+1}(v_1))$$

is linearly dependent in W.

Thus, *any* $N+1$ vectors in W form a linearly dependent list. Hence W contains no linearly independent lists of length greater than N. Hence $\dim W \leq N$. This concludes the proof.