Math 110, Spring 2019.

Homework 2 solutions.

Prob 1. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Counterexample: Here is one of the easist counterexamples: Take V to be any nonzero vector space. Take $U_1 := \{0\}$, $U_2 := V$, $W := V$. Then $U_1 + W = U_2 + W = V$ but $U_1 \neq U_2$.

Prob 2. Suppose
$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$
Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution: $U = \operatorname{span}\{(1, 0, 1, 1, 2), (0, 1, 1, -1, 0)\}$ because a linear combination of these two vectors with coefficients x and y is precisely $(x, y, x + y, x - y, 2x)$. These two vectors are linearly independent because $(x, y, x + y, x - y, 2x) = (0, 0, 0, 0, 0)$ implies its first two components must be zero, i.e., $x = y = 0$. Hence $U = \operatorname{span}\{(1, 0, 1, 1, 2)\} \oplus \operatorname{span}\{(0, 1, 1, -1, 0)\}$.

Let us take $W_1 := \operatorname{span}\{(1, 0, 0, 0, 0)\}$, $W_2 := \operatorname{span}\{(0, 1, 0, 0, 0)\}$, $W_3 :=\{(0, 0, 1, 0, 0)\}$. (These are not the only choices!) The five vectors we now have are linearly independent. Indeed, suppose

$$a_1(1, 0, 1, 1, 2) + a_2(0, 1, 1, -1, 0) + a_3(1, 0, 0, 0, 0) + a_4(0, 1, 0, 0, 0) + a_5(0, 0, 1, 0, 0) = (0, 0, 0, 0, 0).$$

This means

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_2 + a_4 &= 0 \\ a_1 + a_2 + a_5 &= 0 \\ a_1 - a_2 &= 0 \\ 2a_1 &= 0 \end{aligned}$$

The last equality implies $a_1 = 0$, then the first equality implies $a_3 = 0$, then the fourth equality implies $a_2 = 0$, then the third implies $a_5 = 0$, and finally the second equality implies $a_4 = 0$. So all coefficients must be zero for the linear combination to be zero.

Hence the zero vector can be expressed in only one way as a sum of vectors from U, W_1, W_2 and W_3. By the Condition for a direct sum 1.44, we conclude that the sum $U + W_1 + W_2 + W_3$ is direct.

Finally note that the dimension of a (finite) direct sum of finite-dimensional subspaces is equal to the sum of the dimensions. Indeed, 2.43 implies $\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2$, and repeating this argument $k - 1$ times we can obtain $\dim(U_1 \oplus \cdots \oplus U_k) = \dim U_1 + \cdots + \dim U_k$.

Hence the dimension of our direct sum is 5. Hence it is the entire space $\mathbb{F}^5$.

Prob 3. Suppose that the list v_1, v_2, v_3, v_4 is linearly independent in V and that $1 + 1 \neq 0$ in $\mathbb{F}$. Show that the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also linearly independent in V.

Proof: Suppose a linear combination of our new vectors is zero, i.e.,

$$\alpha_1(v_1 - v_2) + \alpha_2(v_1 + v_2) + \alpha_3(v_3 - v_2) + \alpha_4(v_4 - v_1) = 0.$$

This is equivalent to

$$(\alpha_1 + \alpha_2 - \alpha_4)v_1 + (-\alpha_1 + \alpha_2 - \alpha_3)v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0.$$

Since v_1, v_2, v_3, v_4 are linearly independent, this implies

$$\alpha_1 + \alpha_2 - \alpha_4 = -\alpha_1 + \alpha_2 - \alpha_3 = \alpha_3 = \alpha_4 = 0.$$

So, the last two coefficients are zero right away, and $\alpha_1 + \alpha_2 = -\alpha_1 + \alpha_2 = 0$, which implies $2\alpha_1 = 2\alpha_2 = 0$ hence $\alpha_1 = \alpha_2 = 0$ (provided $2 \neq 0$). This concludes the proof of linear independence.

Prob 4. Does the statement of Problem 3 still hold if we replace 'linearly independent' by 'a basis'?

Solution: Yes, again provided, as before, that $1 + 1 \neq 0$ in $\mathbb{F}$. By the new assumption, v_1, v_2, v_3, v_4 form a basis for V, so V is 4-dimensional. By the result of the previous problem, the vectors $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ are linearly independent, and there are 4 of them, so by 2.39, they must form a basis for V.

Prob 5. Prove that the space $\mathbb{R}^{\mathbb{R}}$ is infinite-dimensional.

Remarks: There are literally dozens of ways to prove it. Here are two of the simplest.

Proof 1: The space $\mathbb{R}^{\mathbb{R}}$ contains the inifinite dimensional subspace $\mathcal{P}(X)$, so cannot be finite-dimensional, since the dimension of a finite-dimensional subspace never exceeds the dimension of the entire space by 2.26.

Proof 2: Suppose $\mathbb{R}^{\mathbb{R}}$ has dimension n. By 2.28, the functions $1, x, \ldots, x^{n-1}$ are linearly independent and their number is n, they must form a basis for $\mathbb{R}^{\mathbb{R}}$. But this is false, since, for example, the function x^n is not in their span. Contradiction! So $\mathbb{R}^{\mathbb{R}}$ is infinite-dimensional.

Prob 6. What is the dimension of

(a) $\mathbb{C}$ as a vector space over $\mathbb{C}$?

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$?

(c) $\mathbb{C}^3$ as a vector space over $\mathbb{C}$?

(d) $\mathbb{C}^3$ as a vector space over $\mathbb{R}$?

Solution:

(a) **Answer:** 1**.** The 'standard' basis is $\{1\}$. Indeed, any complex number z is equal to $z \cdot 1$. The singleton set $\{1\}$ is linearly independent since it does not consists of zero. Thus $\{1\}$ is a basis for $\mathbb{C}$ as a vector space over itself.

(b) **Answer:** 2**.** The 'standard' basis is $\{1, \mathrm{i}\}$. Indeed, any complex number $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$ is equal to $a \cdot 1 + b \cdot \mathrm{i}$ so is a linear combination of the vectors 1 and i. These vectors are linearly independent over $\mathbb{R}$ because $a + b\mathrm{i} = 0$ implies $0 = \mathrm{Re}(a + b\mathrm{i}) = a$ and $0 = \mathrm{Im}(a + b\mathrm{i}) = b$. Hence they form a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$.

(c) **Answer:** 3**.** The 'standard' basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Indeed, $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$ and the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are linearly independent since $(x, y, z) = (0, 0, 0)$ implies $x = y = z = 0$. So they form a basis for $\mathbb{C}^3$ as a vector space over $\mathbb{C}$.

(d) **Answer:** 6**.** The 'standard' basis is $\{(1, 0, 0), (\mathrm{i}, 0, 0), (0, 1, 0), (0, \mathrm{i}, 0), (0, 0, 1), (0, 0, \mathrm{i})\}$. Indeed, $(a + b\mathrm{i}, c + d\mathrm{i}, e + f\mathrm{i}) = a(1, 0, 0) + b(\mathrm{i}, 0, 0) + c(0, 1, 0) + d(0, \mathrm{i}, 0) + e(0, 0, 1) + f(0, 0, \mathrm{i})$. If $a, b, c, d, e, f \in \mathbb{R}$, then $(a + b\mathrm{i}, c + d\mathrm{i}, e + f\mathrm{i}) = (0, 0, 0)$ implies $a = b = c = d = e = f = 0$, so these vectors are linearly independent over $\mathbb{R}$. Hence they form a basis for $\mathbb{C}^3$ as a vector space over $\mathbb{R}$.