Math 110, Spring 2019.

Homework 1 solutions.

Prob 1. Suppose $a \in \mathbb{F}$ (field), v, $w \in V$ (vector space over $\mathbb{F}$), and $av = aw$. Prove that $a = 0$ or $v = w$.

Proof. First note that $-(aw) = a(-w)$ since $aw + a(-w) = a(w + (-w)) = a \cdot 0 = 0$, where the last equality is by 1.30. Having proved that $-(aw) = a(-w)$, add $-(aw)$ to both sides of $av = aw$ to obtain

$$0 = aw + (-aw) = av + (-aw) = av + a(-w) = a(v + (-w)) = a(v - w).$$

If the scalar a is zero, the first alternative holds. If $a \neq 0$, there exists its multiplicative inverse $1/a$, hence

$$0 = \frac{1}{a} \cdot 0 = \frac{1}{a}(a(v - w)) = \left(\frac{1}{a} \cdot a\right)(v - w) = 1 \cdot (v - w) = v - w.$$

Here the first equality is due to 1.30, the third due to associativity, and the last due to the multiplicative identity (action) axiom. Hence we obtain $v - w = 0$. Adding w to both sides produces

$$w = (v - w) + w = (v + (-w)) + w = v + ((-w) + w) = v + 0 = v.$$

Thus, $a = 0$ or $v = w$, as we set out to prove.

Prob 2. Let $n \in \mathbb{N}$. Is $\mathbb{Q}^n$ a vector space over $\mathbb{Z}$? Over $\mathbb{Q}$? Over $\mathbb{R}$? Explain.

Solution. $\mathbb{Z}$ does not contain multiplicative inverses of its non-zero elements apart from ± 1 (for example, 2 is not invertible in $\mathbb{Z}$), so $\mathbb{Z}$ is not a field. Hence $\mathbb{Q}^n$ is not a vector space of $\mathbb{Z}$.

Multiplication by scalars from $\mathbb{R}$ does not keep vectors in $\mathbb{Q}^n$ (for example, $\sqrt{2}(1,\ldots,1) = (\sqrt{2},\ldots,\sqrt{2}) \notin \mathbb{Q}^n$), so $\mathbb{Q}^n$ is not a vector space over $\mathbb{R}$.

By 1.10, 1.12, 1.13, 1.14, 1.16, we know that $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ whenever $\mathbb{F}$ is a field. So we will have proven that $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$ once we check that $\mathbb{Q}$ is a field.

Indeed, the operations $+$ and $\cdot$ map rationals to rationals and are commutative, associative, and distributive. Moreover, $0, 1 \in \mathbb{Q}$. Next, $-r \in \mathbb{Q}$ wherenever $r \in \mathbb{Q}$, and $1/r \in \mathbb{Q}$ whenever $r \in \mathbb{Q} \setminus 0$. In other words, $\mathbb{Q}$ contains additive and multiplicative identities, as well as additive inverses of all elements, and multiplicative inverses of all its non-zero elements. Hence $\mathbb{Q}$ is a field, so $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$.

Answers: no, yes, no.

Prob 3. Suppose that $\{0, 1, x\}$ is a field with exactly three elements. What do the addition and multiplication tables *have to be* in that case? Based on the addition and multiplication tables you get, check this is indeed a field. What is the natural way to think of this field (and of x)?

Solution. Since 0 is neutral with respect to addition, we get $0 + 0 = 0$, $0 + 1 = 1$, $0 + x = x$. If $1 + 1 = 1$, then $0 = (-1) + 1 = (-1) + (1 + 1) = ((-1) + 1) + 1 = 0 + 1 = 1$, a contradiction with the requirement $0 \neq 1$. So, $1 + 1 \neq 1$. Likewise, if $1 + x = x$ implies $1 = 0$ and $1 + x = 1$ implies $x = 0$, contrary to the requirement that the elements 0, 1, x are all distinct. Hence $1 + x = 0$. Now, if $1 + 1 = 0 = 1 + x$ would imply $x = 1$, a contradiction again. So $1+1 \neq 0, 1$. Hence $1+1 = x$, and finally $x+x = x+(1+1) = (x+1)+1 = 0+1 = 1$. So the addition table has to be as follows:

$+$	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

Now, 0 times any field element is 0, and 1 times any field element is that element again, so we only need to figure what $x \cdot x$. As $x \neq 0$, x must be invertible, hence $x \cdot x \neq 0$ because $x^{-1} \cdot (x \cdot x) = (x^{-1} \cdot x) \cdot x = 1 \cdot x = x \neq 0$. Likewise $x \cdot x \neq x$ because $x^{-1} \cdot (x \cdot x) = (x^{-1} \cdot x) \cdot x = 1 \cdot x = x \neq 1$. So, $x \cdot x = 1$, and our multiplication table has to look like this:

$*$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

These addition and multiplication tables are exactly the same as the tables in $\mathbb{Z}/3\mathbb{Z}$. As the modular operations $+$ and $\cdot$ satisfy the commutativity, associativity, and distributivity properties and as the tables are the same, our operations satisfy all these properties too.

Finally, directly from the tables we can see that every element has an additive inverse and every non-zero element has a multiplicative inverse. Hence out set with operations $+$ and $\cdot$ is a field, which can be identified with $\mathbb{Z}/3\mathbb{Z}$, and the element x can be identified with 2 (or with -1).

Prob 4. Prove that any field $\mathbb{F}$ is also a vector space over itself, with the field addition used as vector addition, and the field multiplication used as scalar multiplication.

Proof. This is actually a special case of the statement "$\mathbb{F}^n$ is a vector space over $\mathbb{F}$" for $n = 1$. However, we can prove it from scratch as well. Indeed,

- $u + v = v + u$ for any $v,\ v \in \mathbb{F}$ due to the commutativity of addition in $\mathbb{F}$.
- $(u+v)+w = u+(v+w)$ for any $v,\ u,\ w \in \mathbb{F}$ due to the associativity of addition in $\mathbb{F}$, and $(ab)v = a(bv)$ for any $a,\ b \in \mathbb{F}$ due to associativy of multiplication in $\mathbb{F}$.
- $0 \in \mathbb{F}$ is the additive identity in $\mathbb{F}$.
- For every $v \in \mathbb{F}$ there exists its additive inverse in $\mathbb{F}$.
- $1 \in \mathbb{F}$ satisfies $1 \cdot v = v$ for any $v \in \mathbb{F}$.
- Any $v,\ u,\ w,\ \lambda,\ a,\ b \in \mathbb{F}$ satisfy $a(u + v) = au + av$ and $(a + b)u = au + au$ due to distributivity properties in $\mathbb{F}$.

So $\mathbb{F}$ satisfies all axioms of a vector space with respect to addition and mutliplication by scalars from $\mathbb{F}$. Hence $\mathbb{F}$ is a vector space over itself.

Prob 5. For which values of a is the set of all real-valued twice differential functions f on the interval $(0, \infty)$ such that $f''(2) = a$ (equipped with the usual addition of functions and multiplication by real scalars) a vector space over $\mathbb{R}$?

Solution. A necessary condition for forming a vector space is for the set to contain an additive identiity. The additive identity of any function space is the zero function (because addition of that functions is defined pointwise), and all derivatives of the zero function are zero (at any point), so, in particular, $0''(2) = 0$. Hence $a = 0$ is a necessary conndition for our set to form a vector space.

We now show that this condition is sufficient, i.e., that the set

$$V := \{f : (0, \infty) \to \mathbb{R} : \ f'' \text{ exists and } f''(2) = 0\}$$

is a vector space over $\mathbb{R}$. We know from Calculus that the sum of any twice differentiable functions f and g is twice differentiable, and, moreover,

$$(f + g)''(2) = f''(2) + g''(2) = 0 \quad \text{whenever} \qquad f''(2) = g''(2) = 0.$$

Also, for any $\lambda \in \mathbb{F}$, the function λf is again twice differentiable, and

$$(\lambda f)''(2) = \lambda f''(2) = 0.$$

Also, the additive inverse of any function $f \in V$ is again in V because $-f$ is twice differentiable whenever f is and $(-f)''(2) = -f''(2) = 0$.

Finally, addition of real-valued functions is commutative, associative, and distributive with respect to scalars in $\mathbb{R}$, so all the other axioms of a vector space are satisfied. Hence V is a vector space.

Answer: $a = 0$ is necessary and sufficient to form a vector space.