Math 110, Spring 2019.

Homework 11 solutions.

Prob 1. Let $T \in \mathcal{L}(V, W)$. Prove

(a) T is injective if and only if T^* is surjective.

Proof. If T is injective, then $\operatorname{Null} T = \{0\}$, hence $\operatorname{range} T^* = \{0\}^\perp = V$ by 7.7 (b), i.e., T^* is surjective. Conversely, if T^* is surjective, then $\operatorname{range} T^* = V$, hence $\operatorname{Null} T = V^\perp = \{0\}$ by 7.7 (c).

(b) T^* is injective if and only if T is surjective.

Proof. Just replace T by T^*, which replaces T^* by $(T^*)^* = T$ (this equality is due to 7.6(c)), and interchange the spaces V and W in (a) to get (b).

Prob 2. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Proof. Given that $S^* = S$ and $T^* = T$ (i.e., S and T are self-adjoint), we observe that $(ST)^* = T^*S^* = TS$, the first equality by 7.6(e). So $ST = TS$ if and only if $(ST)^* = ST$ whenever S and T are self-adjoint.

Prob 3. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Proof. Suppose P is the orthogonal projector P_U on some subspace U. Then, for any v, $w \in V$, we get

$$\langle P_U v, w\rangle = \langle P_U v, P_U w + (I - P_U)w\rangle = \langle P_U v, P_U w\rangle = \langle P_U v + (I - P_U)v, P_U w\rangle = \langle v, P_U w\rangle. \tag{1}$$

The first equality in (1) is a simple rewrite of w as $Iw = (P_U + (I - P_U))w$, while the second and third use the orthogonality of range P_U and range $(I - P_U)$, and the last equality is a rewrite of v as $Iv = (P_U + (I - P_U))v$. Comparing the leftmost and the rightmost inner products in (1) and recalling that v and w were arbitrary vectors in V, we conclude by 7.11 that P_U is self-adjoint.

Conversely, suppose P is self-adjoint and satisfies $P^2 = P$. Let $U := \operatorname{range} P$. Then, for any v and $w \in V$, we have

$$\langle Pv, (I - P)w\rangle = \langle (I - P)^* Pv, w\rangle = \langle (I - P)Pv, w\rangle = \langle (P - P^2)v, w\rangle = \langle 0v, w\rangle = 0.$$

The first equality here is due to the general properties of the adjoint, the second due to the fact P is self-adjoint, and the second-to-last is due to the fact that $P^2 = P$. Comparing the leftmost and the rightmost inner products and remembering that v and w were arbitrary, we conclude that the range of P and the range of $I - P$ are orthogonal to each other. Hence an arbitrary vector $v \in V$ splits as $Pv + (I - P)v$, where $Pv \in U$ and $(I - P)v \in U^\perp$. So $P = P_U$ by Definition 6.53.

Prob 4. Let $n \in \mathbb{N}$ be fixed. Consider the real space $V := \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx)$ with inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ is *anti-Hermitian*, i.e., satisfies $D^* = -D$.

Proof. First observe that all functions in our space are periodic with period 2π. Hence $f(\pi) = f(-\pi)$ for all $f \in V$. Now, given arbitrary f, $g \in V$, apply integration by parts to the following inner product:

$$\langle Df, g \rangle = \int_{-\pi}^{\pi} f'(x)g(x)dx = f(\pi)g(\pi) - f(\pi)g(\pi) - \int_{-\pi}^{\pi} f(x)g'(x)dx = -\int_{-\pi}^{\pi} f(x)g'(x)dx = \langle f, (-D)g \rangle.$$

This shows that $D^* = -D$, as required.

Prob 5. Let T be a normal operator on V. Evaluate $\|T(v-w)\|$ given that

$$Tv = 2v, \qquad Tw = 3w, \qquad \|v\| = \|w\| = 1.$$

Solution. Since v and w are eigenvectors of T corresponding to different eigenvalues and since T is normal, v and w are orthogonal by 7.22

$$\begin{aligned} \|T(v-w)\|^2 &= \|Tv\|^2 - \langle Tv, Tw\rangle - \langle Tw, Tv\rangle + \|Tw\|^2 \\ &= \|2v\|^2 - \langle 2v, 3w\rangle - \langle 3w, 2v\rangle + \|3w\|^2 \\ &= \|2v\|^2 + 0 + 0 + \|3w\|^2 \;=\; 13, \end{aligned}$$

so $\|T(v-w)\| = \sqrt{13}$.

Prob 6. Suppose T is normal. Prove that, for any $\lambda \in \mathbb{F}$ and any $k \in \mathbb{N}$,

$$\text{Null}\,(T - \lambda I)^k = \text{Null}\,(T - \lambda I). \tag{2}$$

Proof. First note that $(T - \lambda I)^k$ is normal for any $k \in \mathbb{N}$, being of the form $p(T)$ where $p(x) = (x - \lambda)^k$ where T is normal. By 7.21, $\text{Null}\,(T - \lambda I) = \text{Null}\,(T - \lambda I)^*$, hence, by 7.7 (b) and (d),

$$\text{range}\,(T - \lambda I) = (\text{Null}\,(T - \lambda I)^*)^{\perp} = (\text{Null}\,(T - \lambda I))^{\perp} = \text{range}\,(T - \lambda I)^* = \text{range}(\mathrm{T}^* - \overline{\lambda}\mathrm{I}).$$

Now let prove (2) by induction. The induction base $k = 1$ holds trivially. Let $k > 1$ and assume (2) holds for $k - 1$. Consider an arbitrary $v \in \text{Null}\,(T - \lambda I)^k$ and an arbitrary vector $w \in V$. We have

$$\langle (T - \lambda I)^{k-1} v, (T^* - \overline{\lambda} I) w \rangle = \langle (T - \lambda I)^k v, w \rangle = 0.$$

This shows that $(T - \lambda I)^{k-1} v \perp \text{range}\,(T^* - \overline{\lambda} I)$, and we have just proved that the latter is equal to $\text{range}\,(T - \lambda I)$. Taking $(T - \lambda I)^{k-1} v \in \text{range}\,(T - \lambda I)$, we therefore must have

$$0 = \langle (T - \lambda I)^{k-1} v, (T - \lambda I)^{k-1} v \rangle = \|(T - \lambda I)^k\|^2,$$

hence $(T - \lambda I)^{k-1} v = 0$. Since v was an arbitrary vector from $\text{Null}\,(T - \lambda I)^k$, this shows that

$$\text{Null}\,(T - \lambda I)^k \subseteq \text{Null}\,(T - \lambda I)^{k-1}.$$

The reverse inclusion

$$\text{Null}\,(T - \lambda I)^k \supseteq \text{Null}\,(T - \lambda I)^{k-1}$$

holds for any operator T, therefore

$$\text{Null}\,(T - \lambda I)^k = \text{Null}\,(T - \lambda I)^{k-1} = \text{Null}\,(T - \lambda I).$$