Math 110, Spring 2019.

Homework 10 solutions.

Prob 1. Let e_1, …, e_m be an orthonormal list of vectors. Prove that $v \in \operatorname{span}(e_1, \ldots, e_m)$ if and only if

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2. \tag{1}$$

Proof. Suppose $v \in \operatorname{span}(e_1, \ldots, e_m)$. Then $v = \sum_{j=1}^{m} \alpha_j e_j$ for some scalars α_j, $j = 1, \ldots, m$. Taking inner products of both sides with each e_j like in the proof of 6.30, we see that $\alpha_j = \langle v, e_j \rangle$ for all $j = 1, \ldots, m$. Now 6.25 implies

$$\|v\|^2 = \sum_{j=1}^{m} |\alpha_j|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2.$$

Conversely, suppose (1) holds. Note that the vector $w := v - \sum_{j=1}^{m} \langle v, e_j \rangle e_j$ is orthogonal to each vector e_j, hence orthogonal to the entire subspace $\operatorname{span}(e_1, \ldots, e_m)$. In other words, $v = \sum_{j=1}^{m} \langle v, e_j \rangle e_j + w$ where $w \perp \operatorname{span}(e_1, \ldots, e_m)$. By the Pythagorean Theorem,

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2 + \|w\|^2.$$

On the other hand, $\|v\|^2$ equals the first sum alone, according to (1). Hence $\|w\| = 0$, so $w = 0$, and

$$v = \sum_{j=1}^{m} \langle v, e_j \rangle e_j \in \operatorname{span}(e_1, \ldots, e_m).$$

Prob 2. Consider the space $\mathcal{P}_3(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, x^2, x^3$.

Solution. We have $\langle 1, 1\rangle = \int_{-1}^{1} dx = 2$, so the vector/funciton 1 needs to be normalized by $\sqrt{2}$ to give us $f_1(x) = 1/\sqrt{2}$. The vector/function x, being odd, is already orthogonal to 1 (which is even), so only needs to be normalized as well by $\sqrt{2/3}$ since $\int_{-1}^{1} x^2 dx = 2/3$ to produce $f_2(x) = \sqrt{3/2}\, x$. Finally, x^2 (even) is orthogonal to x (odd) but not to 1 (even), so we need to subtract $\langle x^2, 1\rangle 1/\|1\|^2 = 1/3$ to obtain $x^2 - 1/3$ and then normalize by $\sqrt{8/45}$ to obtain $f_3(x) = \sqrt{45/8}(x^2 - 1/3)$ because

$$\int_{-1}^{1} (x^2 - 1/3)^2 dx = \int_{-1}^{1} (x^4 - 2/3x^2 + 1/9)dx = 2/5 - 4/9 + 2/9 = 8/45.$$

Finally, x^3 is already orthogonal to 1 and x^2 (even) but not to x (odd), so x^3 needs to be adjusted first by subtracting $\langle x^3, x\rangle x/\|x\|^2 = 2/5x \cdot 3/2 = 3x/5$ and then normalizing by $\sqrt{8/175}$ because

$$\int_{-1}^{1} (x^3 - 3/5x)^2 dx = \int_{-1}^{1} (x^6 - 6/5x^4 + 9/25x^2)dx = 2/7 - 12/25 + 6/25 = 2/7 - 6/25 = 8/175.$$

Answer: $1/\sqrt{2}$, $\sqrt{3/2}\, x$, $\sqrt{45/8}\,(x^2 - 1/3)$, $\sqrt{175/8}\,(x^3 - 3x/5)$.

Prob 3. Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |1 + 4x - p(x)|^2 dx$$

is as small as possible.

Solution. Note that the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx (= \int_0^1 f(x)\overline{g(x)}dx)$ gives rise to the norm $\|f\|^2 = \int_0^1 f(x)^2 dx (= \int_0^1 |f(x)|^2)$. The problem asks us to minimize the norm, or, alternatively, the square of the norm of the difference between the vector/function $1 + 4x$ and the subspace U defined by the conditions $p(0) = 0$, $p'(0) = 0$.

To achieve that minimization goal, we have to project $1 + 4x$ orthogonally on that subspace. Note that the polynomials in U are exactly those whose constant and linear terms are zero. For $p(x) = ax^2 + bx^3$ to be our norm minimizer, the difference between $p(x)$ and $1 + 4x$ should be orthogonal to x^2 and x^3, a basis of U. Note that this basis *per se* is not orthonormal or even orthogonal, but it does not matter.

So we must have

$$\begin{aligned} 0 &= \int_0^1 (1 + 4x - ax^2 - bx^3)x^2 dx = 1/3 + 1 - a/5 - b/6 \\ 0 &= \int_0^1 (1 + 4x - ax^2 - bx^3)x^3 dx = 1/4 + 4/5 - a/6 - b/7 \end{aligned}$$

which, after some calculation, gives us $a = 39/2$, $b = -77/5$.

Answer: $p(x) = 39x^2/2 - 77x^3/5$.

Prob 4. Consider a complex vector space $V = \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x)$ with an inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

Let U be the subspace of odd functions in V. What is $U^{\perp}$? Find an orthonormal basis for both U and $U^{\perp}$.

Solution: Since the interval of integration is symmetric around 0, the integral of any odd function is zero. Furthermore, since the product of an even and odd function is odd, any even and any odd function are orthogonal with respect to this inner product. Hence

$$U = \operatorname{span}(1, \cos x, \cos 2x), \qquad U^{\perp} = \operatorname{span}(\sin x, \sin 2x).$$

Furthermore, $\int_{-\pi}^{\pi} \sin kx = \int_{-\pi}^{\pi} \cos kx = 0$ for any $k \in \mathbb{N}$, so the angle addition formulas imply

$$\langle \sin kx, \sin \ell x\rangle = \langle \cos kx, \cos \ell x\rangle = 0 \qquad \text{whenever } \; k \neq \ell.$$

Hence the given functions are already orthogonal and just need to be normalized. We have

$$\|1\|^2 = 2\pi, \quad \|\cos kx\|^2 = \|\sin kx\|^2 = \pi \quad \text{for all } \; k \in \mathbb{N},$$

so an orthonormal basis for U is

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}},$$

and an orthonormal baiss for $U^{\perp}$ is

$$\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}.$$

Prob 5. Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if
$$P_U T P_U = T P_U. \tag{2}$$

Proof. Recall that $P_U u = u$ for all $u \in U$ by 6.55(b).

Let U be invariant under T. Then, for any $u \in U$, $Tu \in U$, hence $P_U(Tu) = Tu$. Now, for any $v \in V$, let $u := T_U v (\in U)$. Then
$$P_U T P_U v = P_U T u = Tu = T P_U v.$$
Therefore (2) holds.

Conversely, assume (2) holds and let u be an arbitrary vector in U. Then we have
$$P_U(Tu) = P_U T P_U u = T P_U u = Tu.$$
Since range$P_U = U$ by 6.55(d), this implies $Tu \in U$. Hence U is invariant under T.