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Homework 9, due Apr 7.

Prob 1. Let V be a complex vector space and let $T \in \mathcal{L}(V)$ satisfy $(T-2I)(T+4I)(T-7I)=0$. What possible values can $\lambda \in \mathbb{C}$ take for it to be an eigenvalue of T?

Solution. Given $(T-2I)(T+4I)(T-7I)=0$, applying each side to ANY $v \in V$, we have

$$[(T-2I)(T+4I)(T-7I)](v) = 0(v).$$

This gives by definition of eigenvalue that there exist precisely three unique eigenvectors of T corresponding to three different eigenvalues: $v_1 \in \ker[T-2I]$, $v_2 \in \ker[T+4I]$, $v_3 \in \ker[T-7I]$. That is, for our given T,

$$Tv_1 = 2v_1\ ,\ Tv_2 = -4v_2\ ,\ Tv_3 = 7v_3.$$

From our given information, we can have $\lambda \in \mathbb{C}$ take on values $\{-4, 2, 7\}$ to be an eigenvalue of T.

□

Prob 2. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues.

(a) Prove that every subspace of V invariant under T is either zero or infinite-dimensional.

(b) Give an example of such an operator T on $V := \mathbb{C}^\infty$ with a T-invariant nonzero proper subspace.

Solution. (a) Axler (5.21) states "every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue," or equivalently:

$$(T \in \mathcal{L}(V) \text{ with } \dim V = n, \mathbb{F} = \mathbb{C}) \implies (T \text{ has an eigenvalues})$$

Consider the contrapositive of this statement.

$$(T \text{ has no eigenvalues}) \implies \neg(T \in \mathcal{L}(V) \text{ with } \dim V = n > 0, \mathbb{F} = \mathbb{C})$$

Hence given T has no eigenvalues and $\mathbb{F} = \mathbb{C}$, it must be so that $\dim V \neq n > 0$. This is precisely true when V infinite-dimensional or zero. Then it follows that every subspace of V invariant under T (has an eigenvalue) is either zero or infinite-dimensional (to be consistent with the above).

(b) We are asked to provide an example of a linear operator T (that has no eigenvalues) where a nonzero proper subspace is T-invariant. A canonical example of such an operator is the "right-shift" operator defined $T \in \mathcal{L}(\mathbb{C}^\infty)$ and $z = (z_1, z_2, z_3, \dots) \in \mathbb{C}^\infty$:

$$(z_1, z_2, z_3, \dots) \overset{T}{\longmapsto} (0, z_1, z_2, z_3, \dots)$$

Consider the subset $U \subset \mathbb{C}^\infty$ of tuples with first element 0. In other words, $U := \{z \in \mathbb{C}^\infty | z = (0, z_2, z_3, z_3, \dots), z_i \in \mathbb{C}\}$. Surely U is a nonzero set (for example, $(0, 1, 0, \dots) \in U$). Also, for example, $(1, 0, 0, \dots) \notin U$, so $U \neq \mathbb{C}^\infty$, and U is thus a proper subset. It is a subspace following from linear properties of tuples forming a vector space. Then for all $u \in U$, we have $T(u) \in U$, so subspace U is invariant under T.

However, this operator T has no eigenvalue. To see this, suppose there exists some eigenvalue $\lambda \in \mathbb{C}$ and $z \neq 0$ with $T(z) = \lambda z$. If $\lambda = 0$, then $T(z) = 0 \implies 0 = z_1, z_1 = z_2, \cdots \implies z = 0$, a contradiction to requirement for eigenvalue. If $\lambda \neq 0$, then consider that $T(z) = \lambda z = (\lambda z_1, \lambda z_2, \lambda z_3, \dots) = (0, z_1, z_2, z_3, \dots)$. But $\lambda z_1 = 0$ with $\lambda \neq 0$ means that $z_1 = 0 = \lambda z_2 \implies z_2 = 0 \implies z_3 = 0 \cdots \implies z = 0$, again a contradiction.

$\square$

Prob 3. Let V be a finite-dimensional complex vector space. Prove that $T \in \mathcal{L}(V)$ is diagonalizable if and only if, for all $\lambda \in \mathbb{C}$,

$$\operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I) = V.$$

Solution. We prove the forward ($\Longrightarrow$) direction first.

From T diagonalizable, all non-diagonal entries are zero, so $T - \lambda I$ is diagonalizable for all $\lambda \in \mathbb{C}$. If we have $(T - \lambda I) \in \mathcal{L}(V)$ diagonalizable, then we must have a basis of eigenvectors for finite n-dimensional V. $T - \lambda I$ diagonalizable gives $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, where each U_i ($i \in \{1, 2, \ldots, n\}$) is a 1-dimensional subspace invariant under $T - \lambda I$. That is, for all $u \in U_i$, $[T - \lambda I](u) \in U_i$, and all U_i fall into either the null space or the range of $T - \lambda I$ (and not both, with zero being the only overlap). Hence $\ker[T - \lambda I] \oplus \operatorname{Im}[T - \lambda] = V$ is a direct sum, and we have this equality (span) from $V = \sum_i U_i$.

Now we prove the backwards ($\Longleftarrow$) statement. Suppose we have $\ker[T - \lambda I] \oplus \operatorname{Im}[T - \lambda I] = V$ for all $\lambda \in \mathbb{C}$. For this to be a direct sum (by Axler's definition), $\ker[T - \lambda I]$ and $\operatorname{Im}[T - \lambda I]$ must both be proper subsets of V. Hence $\ker[T - \lambda I] \neq 0$ so $T - \lambda I$ not injective, and $\operatorname{Im}[T - \lambda I] \neq V$ so $T - \lambda I$ not surjective. This is equivalent to T being diagonalizable.

$\square$

Prob 4. Determine whether or not the function taking the pair $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1y_2 + 2x_2y_3 + 3x_3y_1$ is an inner product.

Solution. Consider $u = (1, 2, 3)$ and $v = (-1, -2, -3)$. Then $u, v \in \mathbb{R}^3$ and $\langle u, v\rangle = 1(1)(-2) + 2(2)(-3) + 3(3)(-1) = -2 - 12 - 9 < 0$. But the definition of an inner product requires that an inner product be strictly nonnegative, so this mapping $((x_1, x_2, x_3), (y_1, y_2, y_3)) \mapsto x_1y_2 + 2x_2y_3 + 3x_3y_1$ does NOT give an inner product.

$\square$

Prob 5. Use the dot product to show that the diagonals of a rhombus are perpendicular to each other.

Solution. Recall that a rhombus is defined as a parallelogram with equal sides. Consider the parallelogram created by two vectors $u, v \in \mathbb{R}^2$ with diagonals $u - v$ and $u + v$ (rhombus by setting $|u| = |v|$). If our dot product $(u - v) \cdot (u + v) = 0$, by definition of orthogonal we have their diagonals are perpendicular.

$$\begin{aligned}
(u - v) \cdot (u + v) &= u \cdot (u + v) - v \cdot (u + v) \ \{\text{left additivity of inner product}\} \\
&= \overline{(u + v) \cdot u} - \overline{(u + v) \cdot v} \ \{\text{conjugate symmetry}\} \\
&= (u + v) \cdot u - (u + v) \cdot v \ \{\text{no imaginary part, } \mathbb{F} = \mathbb{R}\} \\
&= [u \cdot u + v \cdot u] - [u \cdot v + v \cdot v] \ \{\text{left additivity of inner product}\} \\
&= [u \cdot u - v \cdot v] + [v \cdot u - u \cdot v] \ \{\text{associativity of addition}\} \\
&= 0 + [v \cdot u - u \cdot v] \ \{|u| = |v| \implies |u|^2 = u \cdot u = v \cdot v = |v|^2\} \\
&= 0 \ \{v \cdot u = \overline{u \cdot v} = u \cdot v\}
\end{aligned}$$

Hence the diagonals of a rhombus are orthogonal (perpendicular). □

This proof could've been one line if we cited that the dot product distributes over additivity.

Prob 6. Prove that, for all complex numbers a_j, b_j, $j = 1, \dots, n$, the following inequality holds:

$$\Big|\sum_{j=1}^{n} a_j \overline{b_j}\Big|^2 \leq \left(\sum_{j=1}^{n} j\,|a_j|^2\right)\left(\sum_{j=1}^{n} \frac{|b_j|^2}{j}\right).$$

Solution. Consider the set U of natural numbers that satisfy the given inequality. For $j = 1$ we have: $|a_1\overline{b_j}|^2 \leq 1|a_1|^2\frac{1}{1}|b_1|^2$ which is true for all $a_j, b_j \in \mathbb{C}$, so $1 \in U$. Assume we have $k \in U$ such that $\Big|\sum_{j=1}^{k} a_j\overline{b_j}\Big|^2 \leq \left(\sum_{j=1}^{k} j\,|a_j|^2\right)\left(\sum_{j=1}^{k} \frac{|b_j|^2}{j}\right)$.

Then consider $k+1$:

$$\begin{aligned}
\Big|\sum_{j=1}^{k+1} a_j\overline{b_j}\Big|^2 &= \Big|\sum_{j=1}^{k} a_j\overline{b_j} + a_{k+1}\overline{b_{k+1}}\Big|^2 \leq \left(\Big|\sum_{j=1}^{k} a_j\overline{b_j}\Big| + |a_{k+1}\overline{b_{k+1}}|\right)^2 \\
&= \Big|\sum_{j=1}^{k} a_j\overline{b_j}\Big|^2 + |a_{k+1}\overline{b_{k+1}}|^2 + 2|a_{k+1}\overline{b_{k+1}}|\Big|\sum_{j=1}^{k} a_j\overline{b_j}\Big| \\
&\leq \left(\sum_{j=1}^{k} j|a_j|^2\right)\left(\sum_{j=1}^{k}\frac{1}{j}|b_j|^2\right) + |a_{k+1}\overline{b_{k+1}}|^2 + 2|a_{k+1}\overline{b_{k+1}}|\Big|\sum_{j=1}^{k} a_j\overline{b_j}\Big| \\
&\leq \left(\sum_{j=1}^{k} j|a_j|^2\right)\left(\sum_{j=1}^{k}\frac{1}{j}|b_j|^2\right) + |a_{k+1}\overline{b_{k+1}}|^2 + 2\Big|\sum_{j=1}^{k} |a_{k+1}|a_j\overline{b_j}|\overline{b_{k+1}}|\Big| \\
&\leq \sum_{j=1}^{k} j|a_j|^2 \sum_{j=1}^{k}\frac{1}{j}|b_j|^2 + |a_{k+1}\overline{b_{k+1}}|^2 + \sum_{j=1}^{k}\frac{j}{k+1}|a_j|^2|b_{k+1}|^2 + \sum_{j=1}^{k}\frac{k+1}{j}|a_{k+1}|^2|b_j|^2 \\
&= \sum_{j=1}^{k} j|a_j|^2 \sum_{j=1}^{k}\frac{1}{j}|b_j|^2 + |a_{k+1}\overline{b_{k+1}}|^2 + \left(\sum_{j=1}^{k} j|a_j|^2\right)\left(\frac{1}{k+1}\right)|b_{k+1}|^2 + \left(\sum_{j=1}^{k}\frac{1}{j}|b_j|^2\right)(k+1)|a_{k+1}|^2 \\
&= \left[\sum_{j=1}^{k} j|a_j|^2 + (k+1)|a_{k+1}|^2\right]\left[\sum_{j=1}^{k}\frac{1}{j}|b_j|^2 + \frac{1}{k+1}|b_{k+1}|^2\right] \\
&= \left(\sum_{j=1}^{k+1} j\,|a_j|^2\right)\left(\sum_{j=1}^{k+1}\frac{|b_j|^2}{j}\right),
\end{aligned}$$

which is our desired inequality replaced with $k+1$. So $k+1 \in U$, and by induction, this gives $U = \{1, 2, 3, \dots\}$ and we are done.

$\square$

Define the following: $a := (a_1, a_2, \ldots, a_n), b := (b_1, b_2, \ldots, b_n)$ with $a, b \in \mathbb{C}^n$. Then our desired inequality is equivalently:

$$\langle a, b\rangle^2 \leq \sum_{j=1}^{n} j a_j \overline{a_j} \sum_{j=1}^{n} \frac{1}{j} b_j \overline{b_j}$$

Cauchy-Schwarz Inequality (6.15 Axler) gives $\langle u, v\rangle \leq |u||v|$. So we have LHS $= \langle a, b\rangle^2 \leq |a|^2|b|^2$.

On the right-hand-side, by the trivial inequality we have $a_j\overline{a_j} = |a_j|^2 \geq 0$ and $b_j\overline{b_j} = |b_j|^2 \geq 0$, so we can provide a lower bound for the two summands:

$$\sum_{j=1}^{n} j a_j \overline{a_j} \geq \sum_{j=1}^{n} a_j \overline{a_j} = \langle a, a\rangle = |a|^2 \quad , \quad \sum_{j=1}^{n} \frac{1}{j} b_j \overline{b_j} \geq \frac{1}{n} \sum_{j=1}^{n} b_j \overline{b_j} = \frac{1}{n}\langle b, b\rangle = \frac{1}{n}|b|^2$$

From these we have:

$$\Big|\sum_{j=1}^{n} a_j \overline{b_j}\Big|^2 = \langle a, b\rangle^2 \leq |a|^2|b|^2$$