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Math 110, Spring 2019.

Homework 8, due Mar 23.

Prob 1. Let T, $S \in \mathcal{L}(V)$ be such that $TS = ST$. Show that range T and null T are invariant under S.

Solution. First we prove that $\text{Im}(T)$ is an invariant subspace under S.

Let $u \in \text{Im}(T)$, so that there exists some $v \in V$ in the preimage with $Tv = u$. If $Su \in \text{Im}(T)$, then $\text{Im}(T)$ invariant under S by definition and we are done. Consider:

$$\begin{aligned} Su &= S[Tv] \text{ (substitution, } u = Tv) \\ &= [ST](v) \text{ (associativity of composition)} \\ &= [TS](v) \text{ (given, } ST = TS) \\ &= T[S(v)] \text{ (where } S(v) \in V) \end{aligned}$$

Therefore $Su \in \text{Im}(T)$.

Next we prove that $\ker(T)$ is invariant under S. Let $u \in \ker(T)$, so that $T(u) = 0$. If $Su \in \ker(T)$, then we have $\ker(T)$ invariant under S and we are done. Consider:

$$\begin{aligned} S(0) = 0 &\implies S[T(u)] = 0 \text{ (linear transformations send 0 to 0)} \\ &\implies [ST](u) = 0 \text{ (associativity of composition)} \\ &\implies [TS](u) = 0 \text{ (given, } ST = TS) \\ &\implies T[S(u)] = 0 \text{ (associativity of composition)} \\ &\implies S(u) \in \ker(T) \text{ (definition of null space)} \end{aligned}$$

□

Prob 2. Let $S, T \in \mathcal{L}(V)$ and suppose S is invertible. Prove that, for any polynomial $p \in \mathcal{P}(\mathbb{F})$,

$$p(STS^{-1}) = S\,p(T)\,S^{-1}.$$

Solution. Recall that by definition of a polynomial, p has some fixed highest degree n. WLOG fix p and n. We can write, for some $a_i \in \mathbb{F}$,

$$\begin{aligned} p(T) &= \sum_{i=0}^{n} a_i T^i \\ &= a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n \end{aligned}$$

Because this equality holds, we can compose S^{-1} on the right of each side (equal functions evaluated on the same input are equal):

$$\begin{aligned} [p(T)](S^{-1}) &= [a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n](S^{-1}) \\ &= [a_0 S^{-1} + a_1 T S^{-1} + a_2 T^2 S^{-1} + \cdots + a_n T^n S^{-1}] \end{aligned}$$

Similarly because this equality holds, we can compose S on the left of each side (evaluating a fixed function S at the same input):

$$\begin{aligned} S([p(T)][S^{-1}]) &= S[a_0 S^{-1} + a_1 T S^{-1} + a_2 T^2 S^{-1} + \cdots + a_n T^n S^{-1}] \\ &= S a_0 S^{-1} + S a_1 T S^{-1} + S a_2 T^2 S^{-1} + \cdots + S a_n T^n S^{-1} \\ &= S a_0 S^{-1} + a_1 S T S^{-1} + a_2 S T^2 S^{-1} + \cdots + a_n S T^n S^{-1} \\ &= \sum_{i=0}^{n} a_i S T^i S^{-1} \end{aligned}$$

Now consider

$$p(STS^{-1}) = \sum_{i=0}^{n} a_i [STS^{-1}]^i$$

By Lemma 1 below, we have $ST^iS^{-1} = [STS^{-1}]^i$, and each of our terms are equal. Recall that we fix p, and as a consequence the coefficients a_i are consistent between the corresponding terms in each summation. So $p(STS^{-1}) = Sp(T)S^{-1}$, as desired.

□

Lemma 1. For any $k \in \mathbb{N}$, and $S, T \in \mathcal{L}(V)$, S invertible, we claim $[STS^{-1}]^k = ST^kS^{-1}$.

Proof. Consider the set $U \subset \mathbb{N}$ for which this statement holds. Surely $[STS^{-1}]^0 = I_V = SI_VS^{-1} = SS^{-1} = I_V$, so $0 \in U$. Assume $k \in U$. If we show $k+1 \in U$, by Peano's induction axiom, we have $U = \mathbb{N}$ and our statement holds for all $k \in \mathbb{N}$. Consider:

$$\begin{aligned} [STS^{-1}]^{k+1} &= [STS^{-1}][STS^{-1}]^k \text{ (recursive definition of the iterate)} \\ &= [STS^{-1}][ST^kS^{-1}] \text{ (assumption } k \in U) \\ &= [ST][S^{-1}S][T^kS^{-1}] \text{ (associativity of composition)} \\ &= [ST][T^kS^{-1}] \text{ (hypothesis } S \text{ invertible, so } S^{-1}S = SS^{-1} = I_V) \\ &= ST^{k+1}S^{-1} \text{(associativity of composition, and definition of iterate } T^{k+1}) \end{aligned}$$

Therefore $k+1 \in U$. By the principle of induction, $U = \mathbb{N}$ and we have our claim proven for all $k = 0, 1, 2, 3, \ldots$ (which are the values possible to which our functions can be empowered via a polynomial). □

Prob 3. Let v be an eigenvector of $T \in \mathcal{L}(V)$ with eigenvalue λ. Show that

$$(T^3 + 3T^2 - 4T + I)v = (\lambda^3 + 3\lambda^2 - 4\lambda + 1)v.$$

How does this observation generalize?

Solution. Let $p \in \mathcal{P}(\mathbb{C})$ be $p(x) = 1x^0 - 4x^1 + 3x^2 + x^3$. Then we can rewrite our given $T^3 + 3T^2 - 4T + I = p(T)$.

Let $U := p(T) = (T^3 + 3T^2 - 4T + I)$, and let $\phi := (\lambda^3 + 3\lambda^2 - 4\lambda + 1)$. Surely $U \in \mathcal{L}(V)$ as it is a linear combination of an operator and its iterates is closed under such operations, and similarly $\phi \in \mathbb{F}$ as a field is closed under scalar multiplication and addition.

Our hypothesis gives $Tv = \lambda v$. Because $v \neq 0$ (definition of eigenvector), we can take p of each side and get $p[Tv] = p[\lambda v] \implies p(T)(v) = p(\lambda)(v)$. We can do this because our polynomial is defined for λ and T, so we can factor out v. Substituting in U and ϕ as defined above, this gives $p(U)v = p(\phi)v$, and so we have

$$(T^3 + 3T^2 - 4T + I)v = p(U)v = p(\phi)v = (\lambda^3 + 3\lambda^2 - 4\lambda + 1)v,$$

which was to be shown.

□

This observation can be generalized to make the following claim: If v is an eigenvector of $T \in \mathcal{L}(V)$ corresponding to $\lambda \in \mathbb{F}$, then for some $k \in \mathbb{N}$,

$$T^k v = \lambda^k v.$$

A proof of this is done in Problem 5, but a copy is provided here for convenience. Moreover, given $Tv = \lambda v$, we can make the claim that a linear combination of iterates of T correspond to the same linear combination of corresponding powers (iterates) of λ. This follows from linearity and I don't have enough coffee or alcohol to rigorously prove this last bit.

Lemma 1. Given $T \in \mathcal{L}(V)$, $v \in V$ eigenvector of T corresponding to λ (in other words, given $Tv = \lambda v$ for $v \neq 0$), we must have $T^k v = \lambda^k v$ for all $k \in \mathbb{N}$.

Proof. We prove with induction. Consider the set $U \subset \mathbb{N}$ for which this property holds true. Surely $T^0 v = I_V v = v$, and $\lambda^0 v = v$, so $T^0 v = \lambda^0 v$ and $0 \in U$. Assume $k \in U$, so that we have $T^k v = \lambda^k v$. Consider:

$$\begin{aligned}
T^{k+1}v &= [TT^k]v(\text{definition of iterate}) \\
&= T[T^k v](\text{associativity of composition}) \\
&= T[\lambda^k v](T^k v = \lambda^k v \text{ by assumption } k \in U) \\
&= \lambda^k T[v](\lambda^k \in \mathbb{F} \text{ and using linearity of } T) \\
&= \lambda^k[\lambda v](Tv = \lambda v \text{ by hypothesis}) \\
&= \lambda^{k+1}(v)(\text{recursive definition of } c^{k+1} := cc^k)
\end{aligned}$$

Therefore $k + 1 \in U$, and by induction we have $U = \mathbb{N}$. □

Prob 4. Let V be a finite-dimensional real vector space and let $T \in \mathcal{L}(V)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\lambda) := \dim \operatorname{range} (T - \lambda I).$$

Which condition on T is equivalent to f being a continuous function?

Solution. Suppose $\dim V = n$. By definition of dimension, f takes on natural number values. Hence for f to be continuous, it must be constant (otherwise discrete "jumps" would cause discontinuity). Furthermore by rank nullity, we have $\dim \operatorname{Im}(T - \lambda I) = \dim V - \dim \ker(T - \lambda I)$, so $f(\lambda) \leq n$. Recall that a transformation $T : V \to V$ can only have at most $\dim V$ distinct eigenvalues, so there must exist some $x \in \mathbb{R}$ where x is not an eigenvalue of T, and $f(x) = n$. So we need $\forall \lambda \in \mathbb{R}$, $f(\lambda) = n$.

We now propose the following: "Function f as defined above is continuous on all $\mathbb{R}$ if and only if T has no (real) eigenvalue."

First we prove the forward statement. Given $f := \dim \operatorname{Im}[T - \lambda I]$ continuous, then as we have reasoned above, by definition of dimension, $f(\lambda) = n$ must be constant for all $\lambda \in \mathbb{R}$. For all $\lambda \in \mathbb{R}$, if $f(\lambda) = n$, then by rank nullity, $\dim \ker[T - \lambda I] = \dim V - f(\lambda) = n - n = 0$. Recall v is an eigenvector of T corresponding to $\lambda \in \mathbb{F} = \mathbb{R}$ if and only if $v \in \ker[T - \lambda I]$ and $v \neq 0$. But the only such $v \in \ker[T - \lambda I]$ is the zero vector. So T has no eigenvalue (recall eigenvalues must be in $\mathbb{F}$).

Next we prove the backwards statement. Given T has no (real) eigenvalue, then by definition of eigenvalue, there is no $\lambda \in \mathbb{R}$ such that for some nonzero $v \in V$ we have $v \in \ker[T - \lambda I]$. So $\ker[T - \lambda I] = \{0\}$, and by rank nullity, for all $\lambda \in \mathbb{R}$, $f(\lambda) = \dim \operatorname{Im}(T - \lambda I) = \dim V - 0 = n$ and is constant (and thus continuous), as desired.

We can extend this argument further, as existence of an eigenvalue implies and is implied by $T - \lambda I$ not bijective (and neither injective nor surjective). This is given in Axler. By the contrapositive, we have that f continuous (which implies no eigenvalue) is equivalent to $(T - \lambda I) \in \mathcal{L}(V)$ being bijective.

□

Prob 5. Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$ is diagonalizable, and all eigenvalues of T are strictly below 1 in absolute value. Given $\varphi \in V'$ and $v \in V$, what is $\lim_{n\to\infty} \varphi(T^n v)$?

Solution. Recall Axler establishes that an operator on V is diagonalizable if and only if V has an eigenbasis (basis consisting of eigenvectors of T). By hypothesis, we have T digonalizable. Let such a basis of V be $v_1, \ldots, v_n$, with $n = \dim V$.

We are given $\varphi \in V^*$ and $v \in V$. By definition of basis, for scalars $c_i \in \mathbb{F} = \mathbb{C}$, we can write

$$v = \sum_{i=1}^{n} c_i v_i.$$

We have $T \in \mathcal{L}(V)$, so any iterate T^n on v ($T^n v$) lives in V, and consequently:

$$\begin{aligned}
\lim_{n\to\infty} \varphi[T^n v] &= \varphi[\lim_{n\to\infty} T^n v] \\
&= \varphi[\lim_{n\to\infty} T^n(\sum_{i=1}^{n} c_i v_i)] \text{ (substituting given } v \text{)} \\
&= \varphi[\lim_{n\to\infty} \sum_{i=1}^{n} c_i T^n v_i] \text{ (linearity of iterated linear operator } T^n) \\
&= \varphi[\lim_{n\to\infty} \sum_{i=1}^{n} c_i [\lambda^n v_i]] \text{ (Lemma 1, } T^k v = \lambda^k v) \\
&= \varphi[\sum_{i=1}^{n} c_i (\lim_{n\to\infty} \lambda^n) v_i] \text{ (applying the limit on each term of the summation)} \\
&= \varphi[\sum_{i=1}^{n} 0] \text{ (} \lim_{n\to\infty}[\lambda^n] = 0 \text{ for } |\lambda| < 1) \\
&= \varphi(0) = 0 \text{ (linear functional must send 0 to 0)}
\end{aligned}$$

□

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&= \lambda^k [\lambda v] (Tv = \lambda v \text{ by hypothesis}) \\
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\end{aligned}$$

Therefore $k + 1 \in U$, and by induction we have $U = \mathbb{N}$. □