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Math 110, Spring 2019.

Homework 7, due Mar 16.

Prob 1. Suppose V is finite-dimensional and U, W are its subpaces. Prove that $(U \cap W)^0 = U^0 + W^0$.

Solution. For this problem let $v \in V, u \in U, w \in W$. We'll (naively) proceed without using the dual basis. Recall that the annihilator U^0 for $U \subset V$ is defined by $U^0 = \{\varphi \in \mathcal{L}(V, \mathbb{F}) : \varphi(u) = 0, \forall u \in U\}$. From 3.105, we have that the annihilator of a subspace is a subspace of a dual space. Additionally, the sum of two subspaces is a subspace, so we have $U^0, W^0, (U \cap W)^0, U^0 + W^0 \subset \mathcal{L}(V, \mathbb{F})$.

We split into two cases: (1) $U \cap W = 0$, and (2) $U \cap W \neq 0$.

Case (1) : $(U \cap W) = 0_V$. By definition of annihilator, surely any linear functional sends 0 to 0, so $(U \cap W)^0 = 0_V^0 = V^*$ in its entirety. Let $\xi \in (U \cap W)^0 = V^*$. For any $v \in V$, $\xi(v) = 0$. Now consider the RHS of our statement. As mentioned earlier, we have $(U^0 + W^0) \subset V^*$ because a sum of subspaces is a subspace. If we show $V^* \subset (U^0 + W^0)$, we are done. In other words we want to show $f \in V^* \implies f \in (U^0 + W^0)$. If $f(u) = 0$ or $f(w) = 0$, then $f \in (U^0 + W^0)$, and we are done. Then fix some f such that for $u \in U, w \in W$, we have $f(u) \neq 0$ and $f(w) \neq 0$.

For any $v \in V$, linear independence gives a unique representation $v = u + x + w$, for some $x \in V$. There exist linear functionals $t_1 \in U^0$ and $t_2 \in W^0$ with

$$t_1(u) := 0 \ , \ t_1(w) := f(w) \ , \ t_1(x) := f(x) \ ; \ t_2(u) := f(u) \ , \ t_2(w) := 0 \ , \ t_2(x) := 0.$$

Then we have $[t_1 + t_2](v) = [t_1 + t_2](u + x + w) = t_1(u) + t_2(u) + t_1(x) + t_2(x) + t_1(w) + t_2(w)$

$$= 0 + f(u) + f(x) + 0 + f(w) + 0 = f(u + x + w) = f(v)$$

$$\implies f = t_1 + t_2 \quad , \forall v$$

Because $U^0 + W^0$ is the addition of sets U^0 and W^0, because $t_1 \in U^0$, $t_2 \in W^0$, and because we have constructed t_1, t_2, then by linearity of addition, we conclude for any $f \in V^*$ we have $f = t_1 + t_2 \in U^0 + W^0$. We have shown $V^* \subset (U^0 + W^0)$ and $(U^0 + W^0) \subset V^*$, so $(U \cap W)^0 = V^* = U^0 + W^0$.

Case (2) : $(U \cap W) \neq 0_V$. For $1 \leq j \leq k \leq n$, let $v_j, \ldots, v_k$ be an ordered basis for $U \cap W$. Let $x \in (U \cap W)$, so we can write for scalars a_i, $x = a_j v_j + \cdots + a_k v_k$. Additionally for all u, w, by linear independence, we can uniquely write $u = u' + x$ and $w = w' + x$, and $v = u' + w' + x$, where $u' \notin W$ and $w' \notin U$.

Let $l \in (U \cap W)^0$ and $r \in (U^0 + U^0)$.

We first prove $(U \cap W)^0 \subset U^0 + W^0$, which is to show $l \in (U^0 + U^0)$. For all $x \in U \cap W$, by hypothesis we have $l(x) = 0$. Consider $l(v) = l(u' + w' + x) = l(u') + l(w') + 0$. Construct $x, y \in V^*$ with

$$x(v) = x(u' + w' + x) := l(u') \ ; \ y(v) = y(u' + w' + x) := l(w').$$

Then $l(v) = x(v) + y(v)$, and $y \in U^0$, $x \in W^0$, so $l = x + y \in (U^0 + W^0)$, as desired.

Secondly we prove $(U^0 + W^0) \subset (U \cap W)^0$, which is to show $r \in (U \cap W)^0$. By hypothesis, $r \in (U^0 + W^0)$ send either all of U or all of W to 0, and $(U \cap W) \subset U$ and $(U \cap W) \subset W$, so r sets all of $(U \cap W)$ to 0, and by definition of annihilator thus $r \in (U \cap W)^0$. We have proven for both cases zero and nonzero $(U \cap W)$, and for each shown $(U \cap W)^0 \subset (U^0 + W^0)$ and $(U^0 + W^0) \subset (U \cap W)^0$, so the two are equal and we are done.

□

Prob 2. Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V,W)$, and $\operatorname{null} T' = \operatorname{span}(\varphi)$ for some $\varphi \in W'$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$. Give an example of such a pair $T \neq 0$, $\varphi \neq 0$ for $V = \mathbb{R}^2$, $W = \mathbb{R}^3$.

Solution. This follows very intuitively. Recall that we have the following definition:

$$T^*(\varphi) := \varphi \circ T$$

We are given $\ker(T^*) = \operatorname{span}(\varphi)$, which means that all linear combinations of φ send to 0 under the map T^*. In other words, $\ker(T^*) = \operatorname{span}(\varphi) \implies T^*(\varphi) = 0$ simply by definition of kernel and span. Our above definition then gives

$$T^*(\varphi) = [\varphi \circ T] = 0$$

This must hold true for all $v \in V$, so

$$[\varphi \circ T](v) = 0$$
$$\implies \varphi[T(v)] = 0$$

$T(v)$ spans the range of T by definition of range, and sends to zero under φ. So we have $\operatorname{Im}(T) \subset \ker(\varphi)$.

It is NOT the case that $\ker(\varphi) \subset \operatorname{Im}(T)$. We don't have enough restrictions for this. We can easily check this by letting $\varphi = 0$ and $T = 0$. Then we have our desired $\varphi(T) = 0 \implies T^*(\varphi) = 0 \implies \ker T^* = \operatorname{span}(\varphi)$. However, we trivially have $\operatorname{Im}(T) = 0$ and $\ker \phi = W$, and $\operatorname{Im}(T) = \ker(\varphi)$.

HOWEVER let us suppose this question meant to restrict $T \neq 0, \varphi \neq 0$, not only in our example. Then to see that $\ker(\varphi) \subset \operatorname{Im}(T)$, suppose we have $\varphi(w) = 0$ for some $w \notin \operatorname{Im}(T)$, $w \in W$. Fix this $w \notin \operatorname{Im}(T)$ (w necessarily nonzero because it is not in the image of T). Fix an ordered basis $w_1, \ldots w_m$ for W. WLOG let $w := a_1 w_1 \notin \operatorname{Im}(T)$ with the property $\varphi[(a_1)(w_1)] = (k_1)(a_1) = 0$. Recall the definition of linear functional $\varphi(w) = \varphi[\sum_{i=1}^m a_i w_i] = \varphi[a_1 w_1 + \sum_{i=2}^m a_i w_i] = k_1 a_1 \sum_{i=2}^m k_i a_i$ for scalars $k_i, a_i \in \mathbb{R}$. This gives us:

$$\varphi(w) = \varphi[a_1 w_1 + \sum_{i=2}^m a_i w_i] = k_1 a_1 + \sum_{i=2}^m k_i$$

Recall $\sum_{i=2}^m a_i w_i \in \operatorname{Im}(T) \implies \sum_{i=2}^m k_i = 0$, and $k_1 a_1 = 0$, $a_1 \neq 0 \implies k_1 = 0$. However, this must hold for any $w \in W$, and $\varphi \neq 0, \varphi(\sum_{i=2}^m a_i w_i) = 0 \implies k_1 \neq 0$, a contradiction. Thus if we have the restrictions $T \neq 0$ and $\varphi \neq 0$, we have that there exists no such $w \notin \operatorname{Im}(T)$ with $\varphi(w) = 0$, and we have $\ker(\varphi) \subset \operatorname{Im}(T)$. We have shown above $\operatorname{Im}(T) \subset \ker(\varphi)$, so we have $\operatorname{Im}(T) = \ker(\varphi)$, as desired.

Then by basis extension, we can uniquely write $w = w' + x$ for some $w' \in \operatorname{Im}(T)$ and $x \notin \operatorname{Im}(T)$, with $w', x \in W$. We then have $\varphi(w) = 0 = \varphi[w' + x] = \varphi[w'] + \varphi[x]$. We have shown above that $w' \in \operatorname{Im}(T) \implies \varphi(w') = 0$. This means we have $\varphi[w] = \varphi[x] = 0$, with $\varphi \neq 0$. However, $\varphi[w'] = 0$ and $\varphi[x]$ for all possible w', x means

$$\implies 0 = 0 + \varphi[x] \implies \varphi[x] = 0$$

□

Now for a concrete example, we are given $V := \mathbb{R}^2, W := \mathbb{R}^3$, and $\ker(T^*) = \operatorname{span}(\varphi) \implies [\varphi(T)] = 0$. Additionally, we need $T, \varphi \neq 0$. Take the standard basis $(1,0),(0,1)$ for $V = \mathbb{R}^2$. Define linear map $V \xrightarrow{T} W$ with

$$(1,0) \overset{T}{\longmapsto} (1,0,0) \;\; ; \;\; (0,1) \overset{T}{\longmapsto} (0,1,0).$$

Let $w \in W$ and take our standard basis for the co-domain $\mathbb{R}^3$, $(1,0,0),(0,1,0),(0,0,1)$. Then for scalars $a,b,c \in \mathbb{R}$ we can write $w = a(1,0,0) + b(0,1,0) + c(0,0,1) = (a,b,c)$.

Define linear functional $\varphi \in W^*$ as $w = (a,b,c) \overset{\varphi}{\longmapsto} c$. Neither T nor φ are zero.

We note that as we've defined T, $\operatorname{Im}(T)$ will always have the form (a,b,c) for some scalars $a,b,c \in \mathbb{R}$ and will always have $c = 0$. All of $\operatorname{Im}(T)$ is in the null space of φ.

It remains to show that the null space of φ is no larger than $\operatorname{Im}(T)$. This is simple in our example because any such $w = (a,b,c) \in W$ with $w \notin \operatorname{Im}(T)$ has $c \neq 0$, and hence $\varphi w = c \neq 0$. So any such $w \notin \operatorname{Im}(T)$ is not in the null space of φ.

Prob 3. Prove that $(\mathcal{P}(\mathbb{R}))^*$ and $\mathbb{R}^\infty$ are isomorphic.

Solution. Let $V^* := \mathcal{P}(\mathbb{R})^*, V := \mathcal{P}(\mathbb{R}), W := \mathbb{R}^\infty$, and $\varphi \in V$.

Consider the morphism of vector addition and scalar multiplication:

$$U : V^* \to W$$

$$f \mapsto (f(1), f(x), f(x^2), f(x^3), f(x^4), \dots)$$

where $f \in V^*$, and $f(p) \in \mathbb{R}$ for $p \in V$. For any $p \in V$, we can write p as a linear combination of our standard monomial infinite basis, for any sequence of scalars a_i:

$$p = \sum_{i=0}^{\infty} (a_i x^i) \ ; \ f(x^i) := a_i$$

To show V^*, W isomorphic, we must show our defined U is injective and surjective.

We have shown before that a linear transformation U is injective if and only if its null space is 0. Take linear functional $f \in \ker(V^*)$, so that $U(f) = (f(1), f(x), f(x^2), \dots) = 0$. This implies $f(1) = f(x) = f(x^2) = f(x^3) = \cdots = 0$. But recall $f(p)$ can be written as a linear combinatoin of these, so $\forall p$, $f(p) = 0 \implies f = 0$. We have shown the only $f \in \ker(V^*)$ is the zero functional itself, and we have U injective.

We have defined $f(x^i) = a_i$, where a_i is an arbitrary (but fixed for that value of i) element of $\mathbb{R}$. Because we can choose a_i to our desire, we have for any $x \in W$, $x = (a_0, a_1, a_2, a_3, \dots) = (f(1), f(x), f(x^2), f(x^3), \dots)$, so $x \in \text{Im}(U)$, so our morphism U is surjective onto all of $\mathbb{R}^\infty$ provided the axiom of choice.

We have shown U is a bijection and this shows $(\mathcal{P}(\mathbb{R}))^*$ and $\mathbb{R}^\infty$ are isomorphic.

$\square$

Prob 4. Prove that every polynomial of odd degree with real coefficients has a real zero.

Solution. WLOG write our polynomial out $p \in \mathcal{P}_{2n+1}(\mathbb{R})$ with $n \in \mathbb{N} \backslash \{0\}$ as $p(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_2x^2 + a_1x + a_0$.

The approach using the intermediate value theorem is perhaps the most intuitive and regular. Instead recall the Fundamental Theorem of Algebra, which states "Every nonconstant polynomial with complex coefficients has a zero." Furthermore, from 4.15, we have that for our polynomial p with real coefficients, "if $\lambda \in \mathbb{C}$ is a zero, then so is $\overline{\lambda}$." In other words, because our coefficients are all real, for every complex root, its conjugate is also a root. We take these to be proven and exhibited in the text.

According to the fundamental theorem of algebra, we have an odd number of roots for our polynomial with odd degree. We let our polynomial be of odd degree $2n + 1$ for some $n \in \mathbb{N} \backslash \{0\}$. So either trivially by definition of odd number or by a pidgeonhole argument subtracting pairs of complex numbers at a time, we are left with one "last" root which must have no imaginary part to ensure the polynomial has real coefficients. We have shown that our polynomial has a real zero, and we are done.

□

Prob 5. Let $p \in \mathcal{P}_n(\mathbb{C})$ for some n and suppose there exist distinct real numbers x_0, x_1, …, x_n such that $p(x_j) \in \mathbb{R}$ for all $j = 0, \ldots, n$. Prove that all coefficients of p are real.

Solution. Given an n-degree polynomial and $n+1$ distinct points $x_0, x_1, \ldots, x_n$, at each of which we have a real value $p(x_j) \in \mathbb{R}$. Using our result in problem 6, for our $n+1$ distinct data points on our domain, we have a unique Lagrange Interpolating Polynomial $f \in \mathcal{P}_n$ (of degree n) that interpolates p at $n+1$ distinct points. Furthermore by definition and construction of polynomial, we have $f = p$.

Recall the set of polynomials $\{L_i\}$:

$$L_i = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} = \prod_{j\neq i}[\frac{x-x_j}{x_i-x_j}]$$

and recall that because we have $n+1$ distinct points interpolating a n-degree polynomial, we can uniquely (again by problem 6) write our given $p \in \mathcal{P}_n(\mathbb{C})$ as a linear combination of these polynomials as follows:

$$f = \sum_{i=0}^{n}[p(x_i)L_i] = \sum_{i=0}^{n}[p(x_i)\frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}]$$

Expanding, we see that all non-x terms in our Lagrange interpolating polynomial p are real-valued, and there are no operations present that could bring rise to an imaginary part. All coefficients of p are multiples and sums of real numbers and are thus real.

□

Prob 6. [Lagrange interpolation.] Prove *using linear algebra*: given distinct *data sites* x_j and arbitrary *data* y_j, $j = 0, \ldots, n$, there is a unique polynomial $p \in \mathcal{P}_n(\mathbb{R})$ such that $p(x_j) = y_j$, for all $j = 0, \ldots, n$.

Solution. We are given $n+1$ distinct data sites, each with an arbitrary but fixed data value. Suppose the data site x_j and data value y_j are both in $\mathbb{R}$. We proceed with Hoffman-Kunze's proof on Lagrange's interpolation formula, with the addition of uniqueness of our polynomial for $n+1$ distinct data sites.

Let V be $\mathcal{P}_n(\mathbb{R})$, the vector space consisting of all polynomials of degree less than or equal to n, including the zero polynomial. Let L_i be the linear functional (linear because $L_i[ap_1 + p_2] = aL_i[p_1] + L_i[p_2]$) from V into $\mathbb{R}$ defined for p in V by

$$L_j(p_i) = p_i(x_j) = y_j \ ; \ 0 \leq i \leq n.$$

Consider the set of $L_0, L_1, L_2, \ldots, L_n$, for which we use Axler's definition of functionals evaluated at different points forming a dual basis to the basis $p_0, p_1, \ldots, p_n$ for $V = \mathcal{P}_n(\mathbb{R})$ (Lemma 1).

Consider then the set of polynomials $\{p_i\}$:

$$p_i = \frac{(x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)} = \prod_{j \neq i} [\frac{x - x_j}{x_i - x_j}]$$

By inspection these polynomials are of degree n and so live in V, so they satisfy our defined linear functional mapping $L_j(p_i) = p_i(x_j)$ above. To show our polynomials p_i are linearly independent in V, suppose $\sum_{j=0}^{n} a_j L_j = 0$. Then this must be so for all data sites x_j. That is, we can use linearity and the definition of dual basis (3.98) as follows:

$$[a_0p_0 + \cdots + a_np_n][x_0] = 0[x_0] \implies [a_0p_0(x_0) + \cdots + a_np_n(x_0) = 0] \implies [a_0 + 0 + \cdots + 0 + 0 = 0] \implies a_0 = 0$$

$$[a_0p_0 + \cdots + a_np_n][x_1] = 0[x_1] \implies [a_0p_0(x_1) + \cdots + a_np_n(x_1) = 0] \implies [0 + a_1 + \cdots + 0 + 0 = 0] \implies a_1 = 0$$

$$\vdots$$

$$[a_0p_0 + \cdots + a_np_n][x_n] = 0[x_n] \implies [a_0p_0(x_n) + \cdots + a_np_n(x_n) = 0] \implies [0 + 0 + \cdots + 0 + a_n = 0] \implies a_n = 0$$

So we have $p_0, p_1, \ldots, p_n$ linearly independent in $V = \mathcal{P}_n(\mathbb{R})$. Because our list p_i has $n + 1 = \dim \mathcal{P}_n(\mathbb{R})$ linearly independent (the correct dimension) polynomials in V, it spans V and is a basis for V. Now for each $p \in V$, consider the "Lagrange interpolating polynomial"

$$p := \sum_{i=0}^{n} y_j p_i.$$

We have shown p_j, each of degreen n to be an ordered basis for $\mathcal{P}_n(\mathbb{R})$ Because we are applying scalar coefficients $y_j = p(x_j)$ to each linearly independent p_j and summing, from linear independence of p_j, the system of $n+1$ equations with $n+1$ unknown has a unique solution. This can also be shown by viewing our interpolation as a linear map that we have shown to be surjective to our space $\mathcal{P}_n(\mathbb{R})$ as we have found a basis, and injective as our vectors are linearly independent before and after the transformation (hence null space is 0).

□

Lemma 1. If p_i (different notation from above) denotes a polynomial with highest degree exactly i, the list $p_0, p_1, p_2, \ldots, p_n$ provides a basis for $\mathcal{P}_n$ over $\mathbb{F}$.

Very shortly, suppose $\sum_{i=0}^{n} a_i p_i = 0$ for scalars a_i. Then surely $a_n = 0$ because it is the only coefficient to the nth degree term. Then it follows $a_n = 0 \implies a_{n-1} = 0 \implies \cdots \implies a_0 = 0$.