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Homework 6, due March 9.

Prob 1. Let V be a vector space over $\mathbb{F}$. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic.

Solution. To show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic, we construct and specify the isomorphism. In the spirit of pi day, define linear transformation

$$U : \mathcal{L}(\mathbb{F}, V) \to V$$

$$f \mapsto f(\pi),$$

for $f \in \mathcal{L}(\mathbb{F}, V)$, with $\pi \approx 3.14159 \in \mathbb{F}, f(\pi) \in V$. If our U is an isomorphism, then we have V and $\mathcal{L}(\mathbb{F}, V)$ isomorphic. From our construction U is a morphism that carries over operations of vector addition and scalar multiplication. Our source $\mathcal{L}(\mathbb{F}, V)$ is a vector space, and our target V is also a vector space. By definition of vector spaces, our constructed U is a morphism. It remains to prove invertibility of our morphism, which we do by showing injectivity and surjectivity.

Injectivity: We have shown before that a linear transformation U is injective iff its null space is 0. Take linear transformation $f \in \ker(U)$, where $f : \mathbb{F} \to V$, so that $U(f) = f(\pi) = 0$ by definition of null space and our define mapping for U. Let $c \in \mathbb{F} \implies f(c) = \frac{c}{\pi} f(\pi) = \frac{c}{\pi} 0 = 0$. So f is necessarily the zero map $f = 0 : \mathbb{F} \to V$, and $\ker(U) = 0$, so U injective.

Surjectivity: To show $U : \mathcal{L}(\mathbb{F}, V) \to V$ surjective, we must show $\forall v \in V$, there exists at least some $u \in U$, $u : \mathcal{L}(\mathbb{F}, V) \to V$ with $u(t) = v$ (for $t : \mathbb{F} \to V$). But we have defined $u(t) = t(\pi) \in V$ at the beginning. We reason that for any $v \in V$, we can construct (there exists) some $t \in \mathcal{L}(\mathbb{F}, V)$ such that t maps some $c \in \mathbb{F}$ to v (that is, $t(c) = v$) (note that for infinite-dimensional V, this would invoke the axiom of choice). Such a linear transformation t lives in $\mathcal{L}(\mathbb{F}, V)$, so in other words, for each v there exist u, t with $v = u(t) = t(c) = \frac{c}{\pi} t(\pi)$, and we have U surjective.

We have shown that U is injective (monomorphism) and surjective (epimorphism), which by definition gives $U : \mathcal{L}(\mathbb{F}, V) \to V$ isomorphic, and we are done.

$\square$

Prob 2. Give an example of V, W, $S \in \mathcal{L}(V,W)$, and $T \in \mathcal{L}(W,V)$ such that

(a) $TS = I$ but S is not invertible.

(b) $TS = I$ but T is not invertible.

Recall that Axler uses the term 'invertible' to describe a transformation which is both left-invertible and right-invertible. Explicitly, his definition is:

Definition: "A linear map $S \in \mathcal{L}(V,W)$ is called **invertible** if there exists a linear map $T \in \mathcal{L}(W,V)$ such that TS equals the identity map on V and ST equals the identity map on W."

It follows that $T \in \mathcal{L}(W,V)$ as defined above is called invertible if there exists $S \in \mathcal{L}(V,W)$ with $TS = I_V$ and $ST = I_W$.

Solution. So we can construct examples where $TS = I_V$ but $ST \neq I_W$, and this would make either S or T not invertible (depending on our construction).

(a) Consider $V := \mathbb{C}^2$ and $W := \mathbb{C}^3$, and fix any bases v_1, v_2 of V, and w_1, w_2, w_3 of W. Define $S : V \to W$ and $T : W \to V$ with the following mappings:

$$v_1 \xmapsto{S} w_1; v_2 \xmapsto{S} w_2$$

$$w_1 \xmapsto{T} v_1; w_2 \xmapsto{T} v_2; w_3 \xmapsto{T} v_1 + v_2$$

Note our designed S is not surjective. Take any $v \in V, w \in W$, and let $a_1, a_2, b_1, b_2, b_3 \in \mathbb{C}$. By definition of basis, we can uniquely write $v = a_1v_1 + a_2v_2$, and $w = b_1w_1 + b_2w_2 + b_3w_3$. Consider $TS(v)$.

$$TS(v) = TS[a_1v_1 + a_2v_2] = T[a_1S(v_1) + a_2S(v_2)] = T[a_1w_1 + a_2w_2]$$

$$= a_1T(w_1) + a_2T(w_2) = a_1v_1 + a_2v_2 = v,$$

so $TS(v) = v \implies TS = I_V$.

Now we show S is not invertible. Consider $w_3 \in W$. If S invertible, then we should have $ST(w_3) = w_3$. But S by our construction maps only to linear combinations of w_1, w_2, and by hypothesis and definition of basis, w_3 is linearly independent from w_1, w_2. So $w_3 \notin \operatorname{Im}(S) \implies ST(w_3) \neq w_3$, and S not invertible.

(b) Now consider $V := \mathbb{C}^2$ and $W := \mathbb{C}^3$, with the any fixed bases v_1, v_2 of V and w_1, w_2, w_3 of W. Then define $S : V \to W$ and $T : W \to V$ with the following mappings:

$$v_1 \xmapsto{S} w_1; v_2 \xmapsto{S} w_2$$

$$w_1 \xmapsto{T} v_1; w_2 \xmapsto{T} v_2; w_3 \xmapsto{T} v_2$$

Note our designed T is not injective. Take any $v \in V, w \in W$, and let $a_1, a_2, b_1, b_2, b_3 \in \mathbb{C}$. Again by definition of basis, we can uniquely write $v = a_1v_1 + a_2v_2$, and $w = b_1w_1 + b_2w_2 + b_3w_3$. Consider $TS(v)$.

$$TS(v) = TS[a_1v_1 + a_2v_2] = T[a_1S(v_1) + a_2S(v_2)] = T[a_1w_1 + a_2w_2]$$

$$= a_1T(w_1) + a_2T(w_2) = a_1v_1 + a_2v_2 = v,$$

so $TS(v) = v \implies TS = I_V$.

To show T is not invertible, it suffices to show for some $w \in W$, $ST(w) \neq w$ because T would fail our definition of invertible. Consider $w_3 \in W$. Per our definitions and mappings, we have:

$$ST(w_3) = S[T(w_3)] = S[v_2] = w_2,$$

but $w_2 \neq w_3$ by hypothesis that w_2, w_3 linearly independent. So T is not invertible.

□

Prob 3. Let $V = \mathcal{P}_2(\mathbb{R})$ and suppose $\varphi_j(p) = p(j)$, $j = 0, 1, 2$. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is a basis for $\mathcal{P}_2(\mathbb{R})'$ and find a basis (p_0, p_1, p_2) of $\mathcal{P}_2(\mathbb{R})$ whose dual is $(\varphi_0, \varphi_1, \varphi_2)$.

Solution. Recall our definition of dual space $\mathcal{P}_2(\mathbb{R})' = \mathcal{P}_2(\mathbb{R})^* = \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathbb{R})$.

We know that $\dim \mathcal{P}_2(\mathbb{R}) = 3 = \dim \mathcal{P}_2(\mathbb{R})^*$. So if we find $\varphi_0, \varphi_1, \varphi_2$ linearly independent in $\mathcal{P}_2(\mathbb{R})^*$ then by matching dimension they must span $\mathcal{P}_2(\mathbb{R})^*$. We've defined for $p \in \mathcal{P}_2(\mathbb{R})$:

$$\varphi_0(p) := p(0); \varphi_1(p) := p(1); \varphi_2(p) := p(2)$$

Suppose we have for $k_0, k_1, k_2 \in \mathbb{R}$, that $k_0\varphi_0 + k_1\varphi_1 + k_2\varphi_2 = 0$. If these are linearly independent, then they must be so for any input p. We evaluate the LHS quantity for $p = 1, x, x^2$ using linearity properties and see if these are sufficient to show $k_0 = k_1 = k_2 = 0$.

$$[k_0\varphi_0 + k_1\varphi_1 + k_2\varphi_2](1) = 0 \implies k_0 + k_1 + k_2 = 0$$

$$[k_0\varphi_0 + k_1\varphi_1 + k_2\varphi_2](x) = 0 \implies 0k_0 + k_1 + 2k_2 = 0$$

$$[k_0\varphi_0 + k_1\varphi_1 + k_2\varphi_2](x^2) = 0 \implies 0k_0 + k_1 + 4k_2 = 0$$

Subtracting the last two equations, we get $k_2 = 0 \implies k_1 = 0 \implies k_0 = 0$, so $\{\varphi_0, \varphi_1, \varphi_2\}$ is a linearly independent list of vectors in $\mathcal{P}_2(\mathbb{R})^*$, with the correct dimension 3, so it is a basis for $\mathcal{P}_2(\mathbb{R})^*$.

We want to find a basis p_0, p_1, p_2 of $\mathcal{P}_2(\mathbb{R})$ such that our dual is $\varphi_0, \varphi_1, \varphi_2$. By definition of dual basis, this gives the following necessary conditions on p_0, p_1, p_2:

$$\varphi_0(p_0) = p_0(0) = 1\ ,\ \varphi_1(p_0) = p_0(1) = 0\ ,\ \varphi_2(p_0) = p_0(2) = 0$$

$$\varphi_0(p_1) = p_1(0) = 0\ ,\ \varphi_1(p_1) = p_1(1) = 1\ ,\ \varphi_2(p_1) = p_1(2) = 0$$

$$\varphi_0(p_2) = p_2(0) = 0\ ,\ \varphi_1(p_2) = p_2(1) = 0\ ,\ \varphi_2(p_2) = p_2(2) = 1$$

We can guess and check the Lagrange interpolating polynomials p_0, p_1, p_2 that satisfies these conditions. (Or equivalently we can just construct a quadratic with any 2 given roots and the third point).

$$p_0 = 1\frac{(x-1)(x-2)}{(0-1)(0-2)} + 0\frac{(x-0)(x-2)}{(1-0)(1-2)} + 0\frac{(x-0)(x-1)}{(2-0)(2-1)} \implies p_0 = \frac{1}{2}(x-1)(x-2)$$

$$p_1 = 1\frac{(x-0)(x-2)}{(1-0)(1-2)} = -(x)(x-2)$$

$$p_2 = 1\frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}(x)(x-1)$$

By our construction (and we can check by plugging in), $\varphi_j(p_k) = 1$ iff $j = k$. It remains to show that our polynomials p_0, p_1, p_2 span $\mathcal{P}_2(\mathbb{R})$. Because our list is of 3 vectors, and $\dim \mathcal{P}_2(\mathbb{R}) = 3$, with the help of dimension, it suffices to show p_0, p_1, p_2 linearly independent to show span.

Let $k_0, k_1, k_2 \in \mathbb{R}$. Suppose $k_0p_0 + k_1p_1 + k_2p_2 = 0$. Then

$$\frac{k_0}{2}[x^2 - 3x + 2] - k_1[x^2 - 2x] + \frac{k_2}{2}[x^2 - x] = 0$$

We can easily see $\frac{2k_0}{2} = k_0 = 0$ because it is the only coefficient to a constant term. Then we have:

$$\implies [\frac{k_2}{2} - k_1]x^2 + [2k_1 - \frac{k_2}{2}]x = 0 \implies [\frac{k_2}{2} - k_1] = 0, [2k_1 - \frac{k_2}{2}] = 0$$

Adding these two equations together, $k_2/2$ cancels out and we get $k_1 = 0 \implies k_2 = 0$. We have shown our p_0, p_1, p_2 linearly independent (and thus a basis) of $\mathcal{P}_2(\mathbb{R})$, and we have dual basis $\varphi_0, \varphi_1, \varphi_2$ as defined in the problem.

□

Prob 4. Let V be a finite-dimensional vector space and let U be its proper subspace (i.e., $U \neq V$). Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$.

Solution. Take $v_1, v_2, \ldots, v_n$ to be our finite-dimensional vector space. We have $U \subset V$ and $U \neq V$, so $\exists v \in V$ such that $v \notin U$. WLOG let such $v = v_1 \notin U$.

We know that a finite-dimensional vector space V has the same dimension as its dual space V^*. We can define a linear functional $\varphi \in V^*$ with $\varphi(v) = \varphi(a_1 v_1, a_2 v_2, \ldots, a_n v_n) := a_1$. Take any $u \in U$, and recall $v_1 \notin U$. Because v_1 and $v_2, v_3, \ldots, v_n$ linearly independent, $u \in U \subset V$ has $a_1 = 0$.

So we have $\varphi(u) = 0$, $\forall u \in U$, and φ is not the zero functional, as desired.

□

Lemma 1. Suppose $v_1, v_2, \ldots, v_n$ ordered basis in V. Then functional $\varphi \in \mathcal{L}(V, \mathbb{F})$ defined as

$$\varphi(v) = \varphi(a_1 v_1, a_2 v_2, \ldots, a_n v_n) = a_1$$

is linear.

Proof. Let $k \in \mathbb{F}$. WLOG consider v_1 and some $v_i, i \neq 1$.

$$\varphi(kv_1 + v_2) = k = k\varphi(v_1) + 0 = k\varphi(v_1) + \varphi(v_2).$$

WLOG consider v_i, v_j, $i, j \neq 1, j \neq i$.

$$\varphi(kv_2 + v_3) = 0 = k\varphi(v_2) + \varphi(v_3).$$

□

Prob 5. Let $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) : p(x) \mapsto (x-1)^3 p(x) + p''(x)$.

(a) Let $\varphi \in \mathcal{P}(\mathbb{R})' : \varphi(p) = p'(1)$. Give a formula for $T'(\varphi)$.

(b) Let $\varphi \in \mathcal{P}(\mathbb{R})' : \varphi(p) = \int_0^1 p(x)\,dx$. Evaluate $T'(\varphi)(x^2)$.

First off I'm not a fan of using T' notation for duality. Confusing as hell in problems like these with differentiation on polynomials. Let's use * instead.

Definition If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T^* \in \mathcal{L}(W^*, V^*)$ defined by $T^*(\varphi) = \varphi \circ T$ for $\varphi \in W^*$.

Solution. (a) We defined $\varphi(p) = p'(1)$. Using our definitions, we have:

$$T^*(\varphi(p(x))) = [\varphi \circ T](p(x)) = \varphi[T(p(x))]$$

$$= \varphi[(x-1)^3 p(x) + p''(x)] = [3(x-1)^2 p(x) + (x-1)^3 p'(x) + p^{(3)}(x)]|_{x=1}$$

$$= [0 + 0 + p^{(3)}(1)] = p^{(3)}(1)$$

(b) We defined $\varphi(p) = \int_0^1 p(x)dx$. Using our definitions, we have:

$$T^*(\varphi)(x^2) = \varphi \circ T(x^2) = \varphi[T(x^2)] = \varphi[(x-1)^3 x^2 + (x^2)'']$$

$$= \int_0^1 [(x-1)^3 x^2 + 2]dx = \int_0^1 [(x^3 - 3x^2 + 3x - 1)(x^2) + 2]dx$$

$$= \int_0^1 [x^5 - 3x^4 + 3x^3 - x^2 + 2]dx$$

$$= [\frac{x^6}{6} - \frac{3x^5}{5} + \frac{3x^4}{4} - \frac{x^3}{3} + 2x]|_0^1$$

$$= \frac{119}{60}$$

□