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**Math 110, Spring 2019.**

# Homework 5, due Mar 2.

**Prob 1.** Let $V := \mathbb{C}^3$. Give an example of a map $T \in \mathcal{L}(V, V)$ such that $V = \operatorname{null} T \oplus \operatorname{range} T$, with both $\operatorname{null} T$ and $\operatorname{range} T$ non-zero, or prove that none exists.

**Solution.** We show an example that holds. Consider the following identity map $T \in \mathcal{L}(\mathbb{C}^3, \mathbb{C}^3)$ defined below, and consider $V$ as a vector space over field $\mathbb{C}$. Take the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ for both our domain and codomain of $T$. (Allowing scalar multiplication by $\alpha \in \mathbb{C}$, we see these vectors span $\mathbb{C}^3$ and are indeed linearly independent.)

$$(1,0,0) \xmapsto{T} (0,0,0)$$

$$(0,1,0) \xmapsto{T} (0,1,0)$$

$$(0,0,1) \xmapsto{T} (0,0,1)$$

Then $(1,0,0)$ is our basis for the null space of $T$, and $(0,1,0),(0,0,1)$ is our basis for the image of $T$. Because these vectors are linearly independent, the two subspaces $\ker(T)$ and $\operatorname{Im}(T)$ have no intersection besides 0, and per Axler's definition of direct sum, $\ker(T) + \operatorname{Im}(T)$ is a direct sum $\oplus$. The direct sum of these two subspaces gives a space described by basis $(1,0,0),(0,1,0),(0,0,1)$, a basis which we have already stated do span $V = \mathbb{C}^3$ under scalar multiplication by scalars in $\mathbb{C}$.

For our constructed $T$, this gives $V = \ker(T) \oplus \operatorname{Im}(T)$, as required.

$\square$

**Prob 2.** Given an example of a map $T \in \mathcal{L}(\mathbb{R}^6, \mathbb{R}^2)$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 = x_2, \;\; x_3 + x_6 = 0, \;\; x_1 + x_3 - x_5 = 0\}$$

or prove that none such exists.

**Solution.** By our definition of $T$, $\operatorname{Im}(T) \subset \mathbb{R}^2$. By rank-nullity, this gives

$$\dim \operatorname{Im}(T) \leq \dim \mathbb{R}^2 \implies \dim \ker(T) \geq \dim \mathbb{R}^6 - \dim \mathbb{R}^2 \implies \dim \ker(T) \geq 4.$$

Consider our requirement for $\ker(T)$, and see we need $x_1 = x_2, x_3 = -x_6, x_5 = x_1 + x_3$. WLOG write $x_1 = x_2, x_3 = -x_6$, and this implies $x_5 = x_2 - x_6$, so we have $x_1, x_3, x_5$ as linear combinations of $x_2, x_4, x_6$. We then have $\dim \ker(T) = 3 = \dim \operatorname{Im}(T)$ as a result of our three requirements defining the set $\ker T$ above. This ($\dim \ker(T) = 3$) contradicts our inequality ($\dim \ker(T) \geq 4$) from rank-nullity above, so such $T$ does not exist.

□

We can construct a basis to verify $\dim \ker(T)$. Let $k \in \ker(T)$. As a result of our requirements, we can write any

$$k = (x_1, x_2, x_3, x_4, x_5, x_6) = (x_2, x_2, -x_6, x_4, x_2 - x_6, x_6)$$

$$= x_2(1, 1, 0, 0, 1, 0) + x_4(0, 0, 0, 1, 0, 0) + x_6(0, 0, -1, 0, -1, 1).$$

Suppose we have $k = 0$. Then we have

$$(0, 0, 0, 0, 0, 0) = x = (a_2, a_2, -a_6, a_4, a_2 - a_6, a_6)$$

$$\implies a_2 = a_4 = 0 \implies a_6 = 0,$$

so our list of vectors $\{(1, 1, 0, 0, 1, 0), (0, 0, 0, 1, 0, 0), (0, 0, -1, 0, -1, 1)\}$ are linearly independent, and by our construction, spans $\ker(T)$ with respect to its requirements. So we have $\dim \ker(T) = 3$.

**Prob 3.** Let $T : f \mapsto f - 2f' + f''$. Write down its matrix representation (a) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using the standard monomial basis both for the domain and codomain; (b) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using the standard monomial basis both for the domain and codomain; (c) as a map in $\mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ using a Newton basis $1, x, x(x-1)$ for the domain and the standard monomial basis for the codomain; (d) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using a shifted monomial basis $1, x-1, (x-1)^2, (x-1)^3$ for the domain and a Newton basis $1, x-1, (x-1)x, (x-1)x(x+1)$ for the codomain.

**Solution.** (a) $1 \overset{T}{\longmapsto} 1$ ; $x \overset{T}{\longmapsto} x - 2$ $\quad x^2 \overset{T}{\longmapsto} x^2 - 4x + 2$

For $\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$ (the elements $1, x, x^2$) as basis for the domain and codomain, our matrix for $T$ is as follows:

$$A_a = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $1 \overset{T}{\longmapsto} 1$ ; $x \overset{T}{\longmapsto} x - 2$ ; $x^2 \overset{T}{\longmapsto} x^2 - 4x + 2$ ; $x^3 \overset{T}{\longmapsto} x^3 - 6x^2 + 6x$

For $\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$ as basis for the domain and codomain, our matrix for $T \in \mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ then is as follows:

$$A_b = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Take $\begin{bmatrix} 1 \\ x \\ x(x-1) \end{bmatrix}$ as our basis for the domain, and $\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$ as our basis for the codomain.

$$1 \overset{T}{\longmapsto} 1 \;;\; x \overset{T}{\longmapsto} x - 2 \;;\; x(x-1) \overset{T}{\longmapsto} x^2 - 5x + 4$$

Our matrix representation of $T \in \mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ is as follows:

$$A_c = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Take $\begin{bmatrix} 1 \\ (x-1) \\ (x-1)^2 \\ (x-1)^3 \end{bmatrix}$ as our basis for the domain, and $\begin{bmatrix} 1 \\ (x-1) \\ (x-1)x \\ (x-1)x(x+1) \end{bmatrix}$ as our basis for the codomain.

For $T \in \mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ and our chosen bases for domain and codomain, we have:

$1 \overset{T}{\longmapsto} 1$ ; $x-1 \overset{T}{\longmapsto} 1(x-1) - 2(1)$ ; $(x-1)^2 \overset{T}{\longmapsto} 1(x-1)(x) - 5(x-1) + 2(1)$ ; $(x-1)^3 \overset{T}{\longmapsto} 1(x-1)(x)(x+1) - 9(x-1)(x) + 13(x-1)$

This gives us the matrix representation of $T$ as follows:

$$A_d = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\square$

**Prob 4.** Suppose $V$ and $W$ are finite-dimensional vector spaces. Let $v \in V$, and consider

$$E := \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$?

**Solution.** (a) $E$ by definition is a subset of $\mathcal{L}(V, W)$, and $0v = 0 \implies 0 \in E$, so $E$ nonempty. We then need to show that elements of $E$ preserve vector addition and scalar multiplication. Consider vectors $e_1, e_2 \in E$, and scalar $a \in \mathbb{F}$.

We need $ae_1 + e_2 \in E$. By our definition of $E$, $e_1$ is some $T_1 \in \mathcal{L}(V, W)$, and $e_2$ is similarly some $T_2$, were $T_1 v = T_2 v = 0$. So we have

$$ae_1 + e_2 = aT_1 + T_2.$$

Consider some $v \in V$.

$$[ae_1 + e_2](v) = [aT_1 + T_2](v) = aT_1(v) + T_2(v) = a(0) + 0 = 0,$$

by our requirement for elements in the set $E$. So we have closure under scalar multiplication and vector addition (of transformations), and $E$ nonempty, so $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) Consider a linear transformation $U : \mathcal{L}(V, W) \to W$ such that for $T \in \mathcal{L}(V, W)$, we have the mapping $T \overset{U}{\mapsto} Tv$. Then $\ker U = E$. We argue that for any $w \in W$, there exists (we can construct) some $T \in \mathcal{L}(V, W)$ that maps all $v \in V$ to $w$ (recall $v \neq 0$). Then $U(T) = T(v) = w$, and $U$ is necessarily surjective onto $W$. This gives $\text{Im}(U) = W$. Then by rank nullity and our constructions $\ker(U) = E$ and $\text{Im}(U) = W$, we have

$$\dim \mathcal{L}(V, W) = \dim \ker U + \dim \text{Im} U = \dim E + \dim W$$

$$\implies \dim E = \dim V \dim W - \dim W.$$

□

Note in our final step we assume we know $\dim \mathcal{L}(V, W) = \dim V \dim W$, for finite-dimensional $V, W$. Although this was not yet explicitly mentioned in lecture before the due date of this assignment, we've shown in lecture that we can construct an bijection between matrices and linear transformations, namely $\mathcal{M}^{\dim W, \dim V} \cong \mathcal{L}(V, W)$, thus creating an isomorphism. The dimension of $\mathcal{M}^{\dim W, \dim V}$ is known by constructing a basis of 1 in each spot and 0 in all others. There are $\dim W \dim V$ such linearly independent matrices, the set of which surely spans $\mathcal{M}^{\dim W, \dim V}$. So we have $\dim \mathcal{L}(V, W) = \dim V \dim W$.

**Prob 5.** Call a matrix representation *optimally sparse* if it consists of zeros and ones only and the number of ones is as small as possible. Let $V = \mathcal{P}_4(\mathbb{R})$ and let $T \in \mathcal{L}(V, V) : f \mapsto f(x+2) - 2f(x+1) + f(x)$.

**Scenario 1:** Given the freedom to use any two bases for $V$ as a domain and a co-domain, does $T$ have an optimally sparse representation? If so, find it. **Scenario 2:** Given the freedom to use any *single* basis for $V$ *both* as a domain and a co-domain, does $T$ have an optimally sparse representation? If so, find it.

**Solution.** For the first scenario with the freedom to take any two bases for the domain and co-domain, take our standard basis for our domain $V = \mathcal{P}_4(\mathbb{R})$, $1, x, x^2, x^3, x^4$. We'll construct a basis for our co-domain and prove that our matrix representation for our transformation $T$ is optimally sparse. Now consider the image of $T$ on each of our standard basis vectors for $V$.

$$1 \overset{T}{\longmapsto} 0 \;\; x \overset{T}{\longmapsto} 0 \;;\; x^2 \overset{T}{\longmapsto} 2 \;;\; x^3 \overset{T}{\longmapsto} 6x+6 \;;\; x^4 \overset{T}{\longmapsto} 12x^2 + 24x + 14$$

For scenario 1, we can pick our standard ordered basis (transpose of) $[1, x, x^2, x^3, x^4]$ for the domain and pick, for example, $\{2, 6x+6, 12x^2+24x+14, x^3, x^4\}$ as our ordered basis for the co-domain. This results in the following matrix representation of $T$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As given by our mappings, we have a 5-dimensional basis for our domain, with two basis vectors $(1, x)$ sending to zero, so $\dim \ker T = 2$. Our matrix has rank 3, so we cannot have any fewer ones in our matrix representation of $T$, and we have an optimally sparse representation, as desired.

Now for the second scenario, with the restriction that our domain and co-domain must share the same basis, we first notice: $x^4 \overset{T}{\longmapsto} 12x^2 + 24x + 14$, and $x^3 \overset{T}{\longmapsto} 6x + 6$, and $x^2 \overset{T}{\longmapsto} 2$. Consider then that $24 \overset{T}{\longmapsto} 0$ and $6x + 6 \overset{T}{\longmapsto} 0$. Then we can have 24 and $6x+6$ as a basis for $\ker(T)$, and $24, 6x+6, 12x^2+24x+14$ as a basis for $\operatorname{Im}(T)$ (by Lemma 1). We can extend our basis for $\operatorname{Im}(T)$ to include $x^3$, and $x^4$ to span $\mathcal{P}_4$.

That is, the ordered basis $\{24, 6x+6, 12x^2+24x+14, x^3, x^4\}$ for both the domain and co-domain.

This gives the following matrix representation for $T$, which is the same optimally sparse representation we have found before.

Again by the same argument, we have that this is the unique optimally sparse representation of $T$.

□

**Lemma 1.** A list of $n+1$ polynomials in $\mathcal{P}_n(\mathbb{F})$ with different highest degrees must span $\mathcal{P}_n(\mathbb{F})$.

WLOG arrange (and name) these $n+1$ polynomials in increasing order of highest degree: $p_0, p_1, p_2, \ldots, p_i, \ldots, p_n$. Suppose $\sum_{i=0}^{n} a_i p_i = 0$. Then necessarily $a_n = 0$, which then implies $a_{n-1} = 0 \implies a_{n-2} = 0 \cdots \implies a_0 = 0$, and we have these $n+1$ polynomials linearly independent. Because we have $n+1$ vectors in a $n+1$ dimensional space $P_n$, our list necessarily spans $P_n(\mathbb{F})$.