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Math 110, Spring 2019.

Homework 4, due Feb 23.

Prob 1. Give, with proof, an example of three linear independent maps from $\mathcal{L}(V, W)$ where $V = W = \mathbb{R}^2$ or prove that no such example exists.

Solution. Take $v = (x, y) \in \mathbb{R}^2$, where $x, y \in \mathbb{R}$. Let us define four particular $T_1, T_2, T_3, T_4 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, written explicitly as:

$$T_1 := (x, y) \mapsto (x, 0)$$
$$T_2 := (x, y) \mapsto (y, 0)$$
$$T_3 := (x, y) \mapsto (0, x)$$
$$T_4 := (x, y) \mapsto (0, y)$$

To see T_1, T_2, T_3, T_4 linearly independent, suppose we have $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $L := a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 = 0$. This is to show that L is the zero map. If we show $a_1 = a_2 = a_3 = a_4$, then we are done. We need $\forall v \in V, [a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4](v) = 0$. We can write any $v \in \mathbb{R}^2$ as $v = (x, y), x, y \in \mathbb{R}$. Consider a fixed $v \neq 0$.

$$[a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4](x, y) = 0$$
$$a_1T_1(x, y) + a_2T_2(x, y) + a_3T_3(x, y) + a_4T_4(x, y) = 0$$
$$(a_1x, 0) + (a_2y, 0) + (0, a_3x) + (0, a_4y) = 0$$
$$(a_1x + a_2y, a_3x + a_4y) = 0$$
$$\implies a_1 = a_2 = a_3 = a_4 = 0$$

So we have four linear independent maps T_1, T_2, T_3, T_4, and any subset of three satisfies our problem.

□

Intuitively we may have noticed our transformations take the form of 2x2 matrices: $T_1 = [1, 0; 0, 0]; T_2 = [0, 1; 0, 0]; T_3 = [0, 0; 1, 0]; T_4 = [0, 0; 0, 1]$. This shows linear independence as well.

Prob 2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Solution. Let $v \in V$. The range of S is the set of all images of $v \in V$, $\{S(v) : v \in V\}$. The kernel or null space of T is the set of all $v \in V$ with $T(v) = 0$.

We have range $S \subset \ker T$. This tells us that S takes any v to $w \in V$, with $w \in \ker T$. Then, we have $T[S(v)] = 0$.

Consider the composite linear map $(ST)^2 = STST$. For any $v_1 \in V$, $STS[T(v_1)] = STS[v_2]$, for some $v_2 \in V$ because S, T are self maps from $V \to V$. $STS[v_2] = ST[v_3]$, for some $v_3 \in \ker T$. Then we have $STST(v_1) = ST[v_3] = S[0]$ because $v_3 \in \ker T$.

Linear transformations send $0 \mapsto 0$; $S(0) = 0$. So we have, $\forall v_1 \in V$, for some $v_2 = T(v_1) \in V$, for some $v_3 = S(v_2) \in \ker T$:

$$STST(v_1) = STS(v_2) = ST(v_3) = S(0) = 0.$$

□

Prob 3. Suppose $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is defined by the formula $Tf = f'' + 3f'$. Check that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ and determine the null space and the range of T.

Solution. The question asks us to check that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$. T is a linear combination of differentiation operations, which are linear, so T is linear. It remains to show that $\mathrm{Im}(T) \subset \mathcal{P}_2(\mathbb{R})$. If we have this, then we have $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$, and we are done for this part.

It suffices to define a transformation $T : V \to W$ by its actions on a basis of V. Take the standard basis $x^3, x^2, x^1, 1$ of $V = \mathcal{P}_3(\mathbb{R})$.

Given in the problem, we have:

$$Tf = 3f' + f''$$

Then we define:

$$T(x^3) \mapsto 9x^2 + 6x, T(x^2) \mapsto 6x + 2, T(x) \mapsto 3, T(1) \mapsto 0$$

We see that $T(x^0)$ maps to 0, so the null space of T is the set of all constant polynomials (where all non-constant coefficients $= 0$). **null space** $\ker(T) = \mathrm{span}\{x^0\}$

Consider the list of $\{T(x^3), T(x^2), T(x)\} = \{9x^2+6x,\ 6x+2,\ 3\}$ spans $\mathrm{Im}(T)$ by definition of the image. We will (unnecessarily) show that this list provides a basis for $\mathcal{P}_2(\mathbb{R})$, which will show $\mathrm{Im}(T) = \mathcal{P}_2(\mathbb{R})$.

Linear Independence:

Suppose we have for some $a, b, c \in \mathbb{R}$, $aT(x^3) + bT(x^2) + cT(x) = 0$. This gives

$$a(9x^2 + 6x) + b(6x + 2) + c(3) = 0$$

$$(9a)x^2 + (6a + 6b)x + (2b + 3c)(1) = 0$$

We have written our expression as a linear combination of the standard basis, so we have a system of linear equations

$$(9a = 0), (6a + 6b = 0), (2b + c = 0)$$

$$\implies a = 0 \implies b = 0 \implies c = 0$$

Which was to be shown to prove $T(x^3), T(x^2), T(x)$ are linearly independent.

Span:

We have a list $\{T(x^3), T(x^2), T(x)\}$ of 3 linearly independent vectors in $\mathrm{Im}(T) \subset \mathcal{P}_2(\mathbb{R})$, so our list necessarily forms a basis for (and thus **span**) $\mathcal{P}_2(\mathbb{R})$, which has dimension 3. It follows then that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$.

$\square$

Alternatively, by the definition of a spanning set, we can write any vector $w \in \mathrm{Im}(T)$ as a linear combination of our list L.

For some $a, b, c \in \mathbb{R}$:

$$\forall w \in \mathrm{Im}(T),\ w = a(9x^2 + 6x) + b(6x + 2) + c(3)$$

$$= (9a)x^2 + (6a + 6b)x + (2b + 3c)$$

We have written w as a linear combination of $x^2, x, 1$, our standard basis for $\mathcal{P}_2(\mathbb{R})$. This shows that the span of $\mathrm{Im}(T)$ is the entirety of $\mathcal{P}_2(\mathbb{R})$.

Prob 4. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Solution. This is to prove that T as a surjective function has a right inverse $S \in \mathcal{L}(W, V)$ with $TS = I_W$. Let $v_1, \ldots, v_n$ be a basis for finite-dimensional V.

We first prove the $\implies$ forward statement: If T surjective, there exists right-inverse S.

Take any element $x \in W$. Then because T is surjective, by definition, $\text{Im}(T) = W$, so to each $w \in W$ there is at least one $v \in V$ with $T(v) = w$. Choose one such v for each w and define $S : W \to V$ by letting $S(w)$ be the chosen v. Then $T(S(w)) = w$, so $TS = I_W$, and S is the desired right inverse.

Now we prove the $\impliedby$ backward statement: If a given T has a right-inverse S, then T is surjective.

By definition of inverse, we have $TS = I_W$, which means that $w = T(S(w))\forall w$, so each $w \in W$ is in the image of T, and T is surjective as required.

$\square$

For fun (!!!) let's do the **injection statement for left-inverses**.

"Suppose $W \neq \emptyset$ is finite-dimensional and $S \in \mathcal{L}(W, V)$. Prove that S is injective if and only if there exists $T \in \mathcal{L}(V, W)$ such that TS is the identity map on W."

Solution . $\implies$:
We show that for injective S, there exists a left-identity map T. For each unique $w \in W$, there is at most one $v \in V$ with $v = S(w)$. Pick some $w_0 \in W$.

Consider one $T \in \mathcal{L}(V, W)$ defined as:

$$T(v) := \text{ that } w \text{ with } S(w) = v, \text{ when } w \in \text{Im}(S),$$
$$:= w_0, \text{otherwise.}$$

This transformation T sends each $S(w)$ "back where it came from," so $T(S(w)) = w, \forall w$. This gives $T(S(w)) = I_W$, so T is a desired left inverse for injective S.

$\impliedby$:
We show that if there exists left inverse $T : V \to W$ for given $S : W \to V$, then S must be injective.

$$T(S(w)) = w, \forall w,$$

so $S(w_1) = S(w_2) \implies w_1 = T(S(w_1)) = T(S(w_2)) = w_2$, and S is injective as desired.

$\square$