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Homework 3, due Feb 15.

Prob 1. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-2}^{2} p(x)\,dx = 0\}$.

(a) Find a basis for U.

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Solution. (a) U is defined by the set of polynomials satisfying our condition $\int_{-2}^{2} p(x)\,dx = 0$.

Take a, b, c, d, e polynomial coefficients $\in \mathbb{R}$. We can write $p = ax^4 + bx^3 + cx^2 + dx + e$. Our condition then gives

$$[\frac{a}{5}x^5 + \frac{b}{4}x^4 + \frac{c}{3}x^3 + \frac{d}{2}x^2 + ex]_{x=-2}^{x=2} = 0$$

$$[\frac{a}{5}(2^5 - -2^5) + \frac{c}{3}(2^3 - -2^3) + e(2 - -2)] = 0$$

$$[\frac{a}{5}(2^6) + \frac{c}{3}(2^4) + e(2^2)] = 0$$

$$[\frac{16}{5}(a) + \frac{4}{3}(c) + e] = 0$$

We see that for even powers, 2^{2k} and $(-2)^{2k}$ are equal and thus subtract out. Note this means that we can have any bx^4 and dx^2 in our defined $p(x)$, and our condition would be satisfied. These two vectors x^4 and x^2 by default satisfy our condition, and will be in our basis for U.

We are left with three vectors in our condition, any pair of which can be independent if the third is redefined as dependent on the chosen pair (given from our nontrivial representation of 0).

WLOG we choose x^5 and x^3 to add to our basis for U and let x^0 be a vector $ex^0 = -\frac{16}{5}(a) + \frac{4}{3}(c)$.

$\therefore$ Our chosen basis for U is $\{x^4, x^3, x^2, x\}$.

(b) To extend our basis in (a) to a basis of $\mathcal{P}_4(\mathbb{R})$, we add x^0 (without any restrictions or conditions) to $\{x^4, x^3, x^2, x\}$. After extension, we have our known standard basis for $\mathcal{P}_4(\mathbb{R})$, $\{x^4, x^3, x^2, x, 1\}$.

(c) We can define a subspace W (dim 1) with $W := \{q \in \mathcal{P}_0(\mathbb{R})\}$. In other words, we take W to be the set of polynomials such that all coefficients are 0, except the constant term. This is a indeed a subspace of $\mathcal{P}_4(\mathbb{R})$, because it is nonempty and is closed under scalar multiplication and vector addition. (This is easily verified, or taken to be known about constants from field $\mathbb{R}$.)

We can see this is indeed a direct sum because the only intersection between U and W is the 0 vector. Suppose this is not the case, and that we have some vector $v \neq 0$ in both U and W. To be in W, the polynomial must be a constant x^0, and to be in U, this constant "polynomial" must have $\int_{-2}^{2} ex^0\,dx = 0 \implies e(2 - (-2)) = 0$. We immediately see this can only be true for $e = 0$.

It remains to show that $U \oplus W$ spans $\mathcal{P}_4$. In part (a), we chose a basis for U, and in (b), we chose a basis for W. From the direct sum we can combine their bases to form a new basis for $U \oplus W$, $\{x^4, x^3, x^2, x, x^0\}$. Let $v \in \mathcal{P}_4(\mathbb{R})$. For every such v, we can uniquely write v as a linear combination of our basis vectors for $U \oplus W$. Thus we have that our constructed W does make for $U \oplus W$ to span $\mathcal{P}_4(\mathbb{R})$.

□

Prob 2. Suppose $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 - w, v_2 - w, \ldots, v_m - w) \geq m - 1.$$

Solution. We have m linearly independent vectors $L_1 := v_1, \ldots, v_m \in V$. We immediately have $\dim(V) \geq m$, because otherwise if $\dim(V) < m$, we would have a basis for V with fewer than m vectors and L_1 would not be linearly independent, which is a contradiction to our hypothesis.

Additionally we are given $w \in V$. We wish to prove that subtracting w from each of the linearly independent vectors L_1 at most decreases the dimension of its span by 1. Define our resulting list of vectors as $L_2 := \{v_1 - w, v_2 - w, \ldots v_m - w\}$.

There are two possibilities, and we can separate them into cases for w: (1) w linearly independent from L_1, and (2) w linearly dependent with L_1.

(1) w is linearly independent from L_1. Let's construct a lemma (included below) that shows for w, v_1, v_2 linearly independent, $(v_1 - w)$ and $(v_2 - w)$ are linearly independent. We apply this lemma for w and each pair of elements in L_1, and we have each item in L_2 linearly independent. L_2 is a list of exactly m linearly independent vectors, and $\dim \operatorname{span}(v_1 - w, v_2 - w, \ldots v_m - w) = m$, and we are done.

(2) w is linearly dependent with L_1. We must show here that w is linearly dependent with only one element of L_1. Suppose not. WLOG suppose we have v_1 and v_2 both linearly dependent with w. Then we can write, for some $a, b \in \mathbb{F}$, $w = av_1$ and $w = bv_2$. By hypothesis, $av_1 + bv_2 = 0 \implies a, b = 0$.

Our supposition that $w = av_1 = bv_2$ (w linearly dependent with v_1, and linearly dependent with v_2) gives us $v_1 = \frac{b}{a}v_2$. This implies v_1 and v_2 are dependent, which is a contradiction to our hypothesis (that v_1 and v_2 are linearly independent).

Thus we cannot have two vectors in L_1 that are linearly dependent with w. It remains to show that w linearly dependent with one vector in L_1 (WLOG v_1) satisfies our statement $\dim \operatorname{span}(L_2) \geq m - 1$.

Let $w = av_1$ as defined above. Then $v_1 - w = v_1 - av_1 = (1 - a)v_1$. If $a \neq 1$, we have $\dim \operatorname{span}(L_2) = \dim \operatorname{span}(L_1) = m \geq m - 1$, and we are done. If $a = 1$, one item in L_2 becomes 0, so the $\dim \operatorname{span}(L_2) = \dim \operatorname{span}(L_1) - 1 = m - 1$, and we are done.

$\square$

Lemma 1. Given some $\{w, v_1, v_2\}$ linearly independent in V, $\{(v_1 - w), (v_2 - w)\}$ must be linearly independent.

Proof. Let $a, b, c, \alpha, \beta, \gamma \in \mathbb{F}$. Given $\{w, v_{1,2}\}$ linearly independent, we have:

$$aw + bv_1 + cv_2 = 0 \implies a = b = c = 0.$$

Consider $\alpha(v_1 - w) + \beta(v_2 - w) = 0$. If this implies $\alpha = \beta = 0$, we are done. Suppose there is a nontrivial representation of 0 ($\alpha \neq 0$ or $\beta \neq 0$). Then we rewrite and regroup:

$$\alpha v_1 + \beta v_2 - (\alpha + \beta)w = 0$$

However, v_1, v_2, w are linearly independent by hypothesis, so by linear independent vectors having only the trivial representation of 0, we have $\alpha = \beta = -(\alpha + \beta) = 0$.
So $\{(v_1 - w), (v_2 - w)\}$ must be linearly independent.

$\square$

Prob 3. Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) \quad = \quad & \dim(U_1) + \dim(U_2) + \dim(U_3) \\ & - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ & + \dim(U_1 \cap U_2 \cap U_3)?\end{aligned}$$

Prove this formula or provide a counterexample.

Solution. Consider the 'inclusion-exclusion formula' for two subspaces, and take this to be true as proven by Axler in 2.43 'Dimension of a Sum': $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$. From 'inclusion-exclusion' on two subspaces on defined subspaces U_1, U_2, U_3, we have:

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$
$$\dim(U_1 + U_3) = \dim(U_1) + \dim(U_3) - \dim(U_1 \cap U_3)$$
$$\dim(U_2 + U_3) = \dim(U_2) + \dim(U_3) - \dim(U_2 \cap U_3)$$

WLOG let $(U_1 + U_2)$ to be one subspace, U_3 the other, and apply the "binary" 'inclusion-exclusion formula'. Consider $(U_1 + U_2) \cap U_3$. From above, we have

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Distributing U_3 over $\cap$, we have

$$\dim[(U_1 + U_2) \cap U_3] = \dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim([U_1 \cap U_2] \cap U_3).$$

Now to prove our original claim, consider the sum $U_1 + U_2 + U_3 = (U_1 + U_2) + U_3$, and we have:

$$\dim[(U_1 + U_2) + U_3] = \dim(U_1 + U_2) + \dim(U_3) - \dim[(U_1 + U_2) \cap U_3]$$
$$= [\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)] + \dim(U_3) - [\dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3)]$$
$$= [\dim(U_1) + \dim(U_2) + \dim(U_3)] - [\dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2)] + \dim(U_1 \cap U_2 \cap U_3),$$

which was to be shown.

Edit: This turned out to be wrong, look at stackoverflow if you care for a classic counterexample. Quite certain people who got this problem correct just googled it :)

□

Prob 4. Let $a, b \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(4) + 5p'(2) + ap(1)p(3), \int_{-1}^{2} x^3 p(x)\, dx + b \cos p(0)).$$

Show that T is linear if and only if $a = b = 0$.

Solution. For this problem suppose we have $p_1(x), p_2(x) \in \mathcal{P}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. First we prove the forward statement. Assume we know T is linear, and we show this implies $a = b = 0$.

Given T is linear, we must have $T(0_{\mathcal{P}(\mathbb{R})}) = 0_{\mathbb{R}^2}$. Checking this, we have

$$T(0) = (2(0) + 5(0) + a(0)(0), 0 + b\cos(0)) = (0, b),$$

so we have shown if T is linear, then $b = 0$. Now let's consider a, using $b = 0$ to lessen the clutter.

Let $p_3 = [\lambda p_1 + p_2]$. For T to be linear, we need $T(\lambda p_1 + p_2) = \lambda T(p_1) + T(p_2)$.
First consider LHS:

$$T(\lambda p_1 + p_2) = T(p_3) = (2p_3(4) + 5p_3'(2) + ap_3(1)p_3(3), \int_{-1}^{2} x^3 p_3(x)\, dx)$$

$$= (2[\lambda p_1(4) + p_2(4)] + 5[\lambda p_1'(2) + p_2'(2)] + a[\lambda p_1(1) + p_2(1)][\lambda p_1(3) + p_2(3)], \int_{-1}^{2} x^3[\lambda p_1(x) + p_2(x)]dx)$$

$$= (\cdots + a\lambda^2 p_1(1)p_1(3) + p_1(3)p_2(1) + \lambda p_1(1)p_2(3) + p_2(1)p_2(3), \int_{-1}^{2} x^3[\lambda p_1(x) + p_2(x)]dx)$$

Now consider RHS:

$$\lambda[T(p_1)] + T(p_2) = (\lambda)[(2p_1(4) + 5p_1'(2) + ap_1(1)p_1(3), \int_{-1}^{2} x^3 p_1(x)\, dx)] + (2p_2(4) + 5p_2'(2) + ap_2(1)p_2(3), \int_{-1}^{2} x^3 p_2(x)\, dx)$$

$$= (\cdots + ap_2(1)p_2(3) + a\lambda p_1(1)p_1(3), \int_{-1}^{2} x^3 p_2(x)\, dx + \lambda \int_{-1}^{2} x^3 p_1(x)\, dx)$$

In LHS, for example, we have a term $a\lambda^2 p_1(1)p_1(3)$ which does not exist (impossible to obtain) in RHS. Given T is linear, to equate LHS = RHS, we need $a = 0$. Thus we have proven (T linear $\implies a, b = 0$).

It remains to prove the converse. If $a, b = 0$, then T as defined above is linear. For the purpose of proving this converse, consider the 'new' definition of T with the condition $a, b = 0$.

$$Tp = (2p(4) + 5p'(2), \int_{-1}^{2} x^3 p(x)\, dx)$$

We check two conditions to prove T linear : (1) $T(0) = 0$, and (2) $T(\lambda p_1 + p_2) = \lambda T(p_1) + T(p_2)$ for $p_1, p_2 \in \mathcal{P}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

(1) Let $p_0(x) = 0$. $T(0) = T(p_0) = (2p_0(4) + 5p_0'(2), \int_{-1}^{2} x^3 p_0(x)\, dx) = (0, 0)$, and we are done.
(2) **RHS** $= \lambda T(p_1) + T(p_2) = \lambda(2p_1(4) + 5p_1'(2), \int_{-1}^{2} x^3 p_1(x)\, dx) + (2p_2(4) + 5p_2'(2), \int_{-1}^{2} x^3 p_2(x)\, dx)$
$= (2\lambda p_1(4) + 5p_1'(2) + 2p_2(4) + 5p_2'(2)\ ,\ \lambda \int_{-1}^{2} x^3 p_1(x)\, dx + \int_{-1}^{2} x^3 p_2(x)\, dx)$
LHS $= T(\lambda p_1 + p_2) = (2[\lambda p_1(4) + p_2(4)] + 5[\lambda p_1'(2) + p_2'(2)], \lambda \int_{-1}^{2} x^3 p_1(x)\, dx) + \int_{-1}^{2} x^3 p_2(x)\, dx)$
If desired, we can show that the sum of integrals of those polynomials is equal to the integral of the sum of the polynomials, as they share the same bounds, and the x^3 can be factored out. But we have shown RHS = LHS and verified that T is linear, given $a = b = 0$, and we are done.

$\square$

Prob 5. Suppose $T \in \mathcal{L}(V, W)$, $v_1, \ldots, v_m \in V$ and the list $Tv_1, Tv_2, \ldots Tv_m$ is linearly independent (in W). Prove that $v_1, \ldots, v_m$ must be linearly independent in V. What is the contrapositive of this statement?

Solution. Our statement is: If $Tv_1, Tv_2, \ldots, Tv_m$ is linearly independent in W, then $v_1, v_2, \ldots v_m$ must be linearly independent in V.

The contrapositive of this statement is: If $v_1, v_2, \ldots, v_m$ is NOT linearly independent in V, then $Tv_1, Tv_2, \ldots, Tv_m$ CANNOT be linearly independent in V. By properties of the contrapositive, these statements are logically equivalent.

Given $v_1, v_2, \ldots, v_m$ linearly dependent, then there exists some $i : 1 \leq i \leq m$ where $v_i = \sum_{j \neq i} a_j v_j$. That is, we can write an item v_i as a linear combination of the other $v_{j \neq i}$ in that list.

Then, we have $T(v_i) = T(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j T(v_j)$ in W. We have written $T(v_i)$ as a linear combination of $T(v_j)_{j \neq i}$, so the list

$$Tv_1, Tv_2, \ldots, Tv_m$$

cannot be linearly independent, and we are done. $\square$

Prob 6. Suppose V is a nonzero finite-dimensional vector space and W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution. Let $v_1, v_2, \ldots, v_n$ be a basis for V. That is, let $\dim V = n$ for some finite n. If we find an infinite subset of $\mathcal{L}(V, W)$, then $\mathcal{L}(V, W)$ cannot be finite while containing an infinite subset, and we would be done. We construct such a subset.

Let us begin by introducing a procedure to derive a basis for W. We can pick $w_1 \in W$ because W is nonempty by hypothesis (empty would imply finite dimension). We are given W infinite dimensional. So we know $w_1 \neq W$ (we cannot span W with w_1 alone). So we can pick $w_2 \in W$ such that $w_2 \neq w_1$. Again by hypothesis, W is infinite dimensional, so w_1, w_2 cannot span W. This procedure continues indefinitely, generating a list $w_1, w_2, \ldots$. Because W is infinite-dimensional, this list cannot be finite.

Now consider $\{T_1, T_2, \ldots\} \subset \mathcal{L}(V, W)$, defined T as

$$T_i(v) := w_i, \quad \forall v \in \{v_1, v_2, \ldots, v_n\}, \;\; w_i \in \{w_1, w_2, \ldots\}, \;\; i = 1, 2, \ldots$$

Essentially for each linear map T_i, all $v \in \{v_1, v_2, \ldots, v_n\}$ map to a particular w_i, and to a different $w_{k \neq i}$ for each different transformation $T_{k \neq i}$. To see that T_i is linear, take any two vectors v_1, v_2 from our basis for V, and scalar $c \in \mathbb{F}$.

$$\begin{aligned} T_i(cv_1 + v_2) &= T_i(cv_1 + v_1) = (c+1)T_i(v_1) = (c+1)w_i \\ &= cw_i + w_i = c(T_i v_1) + T_i v_2 \end{aligned}$$

By construction, we then have $\{T_1, T_2, T_3, \ldots\}$ independent, because $\{w_1, w_2, \ldots\}$ independent. Because our list $\{w_1, w_2, \ldots\}$ proceeds infinitely, so does $\{T_1, T_2, T_3, \ldots\}$.

We have found a linearly independent, infinite list $\{T_1, T_2, T_3, \ldots\} \subset \mathcal{L}(V, W)$, so our containing set $\mathcal{L}(V, W)$ cannot be finite-dimensional.

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