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Math 110, Spring 2019.

Homework 2, due Feb 8.

Prob 1. Prove or disprove (i.e., provide a counterexample): if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. We disprove by counterexample. Take $U_1 = \{0\}$, $U_2 = V$, $W = V$.
Then we have $U_1 + W = \{0\} + V = V + V = U_2 + W$, so $U_1 + W = U_2 + W$, but $U_1 \neq U_2$. $\square$

Prob 2. Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Let $\mathbb{F} := \mathbb{C}$ and $x, y \in \mathbb{F}$. Additionally let $p, q, r \in \mathbb{F}$. Suppose we define:

$$W_1 = (0, 0, p, 0, 0) \in \mathbb{F}^5 : p \in \mathbb{F},$$

$$W_2 = (0, 0, 0, q, 0) \in \mathbb{F}^5 : p \in \mathbb{F},$$

$$W_3 = (0, 0, 0, 0, r) \in \mathbb{F}^5 : p \in \mathbb{F}$$

First we quickly verify that U, W_1, W_2, W_3 are all subspaces of $\mathbb{F}^5$. Note that each of these spaces are nonempty by definition. Then it remains to show closure under scalar multiplication and vector addition. Take $c \in \mathbb{C}$; $x_u, y_u \in U$; $x_{w_1}, y_{w_1} \in W_1$; $x_{w_2}, y_{w_2} \in W_2$; $x_{w_3}, y_{w_3} \in W_3$. Then we have:

- $cx_u + y_u \in U$
- $cx_{w_1} + y_{w_1} \in W_1$
- $cx_{w_2} + y_{w_2} \in W_2$
- $cx_{w_3} + y_{w_3} \in W_3$

Thus we have subspaces U, W_1, W_2, W_3. Now we prove that their vectors are linearly independent and that their span is equal to $\mathbb{F}^5$.

By inspection, the i^{th} element of the n-tuple for $i = 1, 2, \ldots, n$ is generated uniquely from no more than one of U, W_1, W_2, W_3. It follows that the basis vectors of these subspaces are all linearly independent. Shown explicitly, we have, for $a, b, c, d, e \in \mathbb{C}$:

$$(ax, by, ax + by + cp, ax - by + dq, 2ax + er) = (0, 0, 0, 0, 0)$$

$$\implies (a = 0), (b = 0), (0x + 0y + cp = 0), (0x - 0y + dq = 0), (2(0)x + er = 0)$$

$$\implies (c = 0), (d = 0), (e = 0)$$

Because the only representation of 0 is the trivial representation $a = b = c = d = e = 0$, we have linear independence, which also tells us that the sum $U + W_1 + W_2 + W_3$ is a direct sum.

It remains to show $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$ by verifying the span of these 5-tuples. Let $a, b, c, d, e \in \mathbb{F}$ and express arbitrary element $x := (a, b, c, d, e) \in \mathbb{F}^5$. Surely we can **uniquely** express $x = (a, b, c, d, e) = (x, y, x + y + p, x - y + q, 2x + r)$, because given fixed x, y we get unique values p, q, r to match given values c, d, e respectively. So, the span of bases vectors from U, W_1, W_2, W_3 is the span of $\mathbb{F}^5$.

Proving linear independence, direct sum, and span, we have $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

$\square$

Prob 3. Let V be a vector space over $\mathbb{F}$. Suppose that $2 \neq 0$ in $\mathbb{F}$ and the list v_1, v_2, v_3, v_4 is linearly independent in V. Show that the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also linearly independent in V.

(Better) Solution A. Perhaps more "explicitly" than our second solution, we can suppose for contradiction that $L_3 := v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is linearly dependent in V. Then, for some $a, b, c, d \in \mathbb{F}$, $\exists$ non-trivial solution to

$$a(v_1 - v_2) + b(v_1 + v_2) + c(v_3 - v_2) + d(v_4 - v_1) = 0$$

(namely, a, b, c, d need not all be 0). Expanding and regrouping, we see:

$$(a + b - d)v_1 + (b - a - c)v_2 + (c)v_3 + (d)v_4 = 0$$

We have expressed L_3 as a linear combination of L_1, which by hypothesis, is linearly independent. Thus $(a + b - d) = (b - a - c) = c = d = 0 \implies (a + b - 0) = (b - a - 0) = 0 \implies a = b = 0$, and we have only the trivial solution $a = b = c = d = 0$. This is a contradiction to our initial supposition, and L_3 must be linearly independent in V.

□

Lemma 1. Given some $\{u_1, u_2\}$ linearly independent in V, $\{u_1, (u_1 - u_2)\}$ must be linearly independent.

Proof. Let $a, b, c, d \in \mathbb{F}$. Given $\{u_1, u_2\}$ linearly independent, we have:

$$au_1 + bu_2 = 0 \implies a = b = 0.$$

Consider $cu_1 + d(u_1 + u_2) = 0 \implies (c + d)u_1 + (d)u_2 = 0 \implies c + d = d = 0 \implies c = 0$
$\therefore \{u_1, (u_1 + u_2)\}$ is a linearly independent list.

□

(My first intuitive) Solution B. Given linearly independent list $L_1 := v_1, v_2, v_3, v_4$ in our hypothesis, we can construct a linearly independent list L_2 by adding v_1 to the second item in L_1 and subtracting v_1 from the fourth item of L_1. By hypothesis v_1, v_2, v_3, v_4 linearly independent and Lemma 1, we have independent L_2:

$$L_2 := v_1, v_1 + v_2, v_3, v_4 - v_1.$$

Similarly, we can subtract v_2 from the first and third items of L_2, both of which are linearly independent to v_2 which we are subtracting, resulting in

$$L_3 := v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1.$$

By Lemma 1, it follows that L_3 is a list linear independent in V.

□

Prob 4. Does the statement of Problem 3 still hold if we replace 'linearly independent' by 'a basis'?

We rewrite the problem here: Let V be a vector space over $\mathbb{F}$. Suppose that $2 \neq 0$ in $\mathbb{F}$ and the list v_1, v_2, v_3, v_4 is **a basis of** V. Show that the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also **a basis of** V."

Solution. Yes, the statement still holds.

Given basis $L_1 := v_1, v_2, v_3, v_4$ of V, we must show that $L_3 := v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also a basis of V. We must prove linear independence of L_3 and span of L_3. We use the result from problem 3 to prove linear independence of L_3. It only remains to prove $\text{span}(L_3) = V$ in this proof.

From our hypothesis, $L_1 = v_1, v_2, v_3, v_4$ provides a basis for and thus spans V, so we have for some $a_1, b_1, c_1, d_1 \in \mathbb{F}$, $\forall x \in V$,

$$x = a_1 v_1 + b_1 v_2 + c_1 v_3 + d_1 v_4.$$

If we can rewrite L_3 as a linear combination of $L_1 = v_1, v_2, v_3, v_4$, we are done.

$$L_3 = a_3(v_1 - v_2) + b_3(v_1 + v_2) + c_3(v_3 - v_2) + d_3(v_4 - v_1)$$

and expand and regroup to have:

$$L_3 = (a_3 + b_3 - d_3)v_1 + (b_3 - a_3 - c_3)v_2 + (c_3)v_3 + (d_3)v_4.$$

As we have written L_3 as a unique linear combination of vectors from L_1, let

$$(a_1 := a_3 + b_3 - d_3), (b_1 := b_3 - a_3 - c_3), (c_1 := c_3), (d_1 := d_3).$$

We see this assignment is unique because for any given a_1, b_1, c_1, d_1, we have:
$d_3 = d_1$, $c_3 = c_1$; (I) $a_1 = a_3 + b_3 - d_1$; (II) $b_1 = b_3 - a_3 - c_1$.

$$(I) + (II) \implies b_3 = \frac{1}{2}(a_1 + b_1 - d_1 - c_1)$$

$$(I) - (II) \implies a_3 = \frac{1}{2}(a_1 - b_1 - c_1 + d_1)$$

We have proven for any given a_1, b_1, c_1, d_1, we have a unique a_3, b_3, c_3, d_3. As we can write any vector $x \in V$ uniquely as a linear combination of L_1 (by hypothesis), and uniquely as a linear combination of L_3 (as proven above), it follows that L_3 spans V. Combined with the results from problem 3, $L_3 = v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is shown to indeed be a basis of V.

□

Prob 5. Prove that the space $\mathbb{C}^R$ is infinite-dimensional.

Solution. This proof uses Hoffman-Kunze's example of an infinite basis (and uses it as a "counterexample" against $\mathbb{C}^{\mathbb{R}}$ being finite-dimensional).

Take $\mathbb{C}^R$ to be a typo for $\mathbb{C}^{\mathbb{R}} := \{f : \mathbb{R} \to \mathbb{C}\}$. That is, we are interested in the set of all functions taking $\mathbb{R}$ to $\mathbb{C}$. If a vector space is not finite-dimensional, then it must be infinite-dimensional.

Suppose for contradiction sake, that the statement is false, and that $\mathbb{C}^{\mathbb{R}}$ is finite-dimensional. Then any given subspace of C^R must be finite-dimensional, else C^R cannot be finite-dimensional. If our subspace of $\mathbb{C}^{\mathbb{R}}$ is infinite-dimensional, then $\mathbb{C}^{\mathbb{R}}$ itself must be infinite-dimensional, and we are done.

Consider V, the space of polynomial functions over $\mathbb{R}$ to be a subspace of $\mathbb{C}^{\mathbb{R}}$. Let $c_i \in \mathbb{C}, i = 0, 1, 2, \ldots$. This space

$$V := \{f : f(x) = c_0 + c_1 x + \cdots + c_n x^n\}$$

is a known subspace of $\mathbb{C}^{\mathbb{R}}$. This is easily shown because scalar multiplication and vector addition are enclosed in V, and the set contains the function $f = 0$ and is thus nonempty.

Let $f^k(x) = x^k$ for $k = 0, 1, 2, \ldots$. Then, propose $\{f_0, f_1, f_2, \ldots\}$ as a basis for V. We see that it spans V because $f = c_0 f_0 + c_1 f_2 + \cdots + c_n f_n$ for any natural number. It remains to show that for every n, $\{f_0, \ldots, f_n\}$ is independent. Suppose $c_0 f_0 + \cdots + c_n f_n = 0$. This by our construction of the polynomial means $c_0 + c_1 x + \cdots + c_n x^n$. Every x in V is a root of the polynomial $f(x) = c_0 + c_1 x + \cdots + c_n x^n$. By a result of the Fundamental Theorem of Algebra, a polynomial of degree n with complex coefficients cannot have more than n distinct roots. This guarantees $c_0 = c_1 = \cdots = c_n = 0$. As this set spans V and is linearly independent, this set is a basis. However, this set clearly cannot be finite, because for any given n, $f_{n+1} \in V$ and $f_{n+1} \notin \{f_0, \ldots, f_n\}$. Therefore, we have an infinite dimension basis for V.

It remains to show that V is infinite-dimensional from the fact that its basis is infinite dimensional. Suppose that we have a finite number n of polynomial functions $g_1, g_2, \ldots, g_n$. Similar to the above argument, in those polynomials, take x^k to be the highest power of x with nonzero coefficient. Then, x^{k+1} is clearly not in the span of $g_1, g_2, \ldots, g_n$. So, V cannot be finite-dimensional.

As mentioned above, as V is a subspace of $\mathbb{C}^{\mathbb{R}}$, and V is not finite-dimensional, then $\mathbb{C}^{\mathbb{R}}$ cannot be finite dimensional.

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Prob 6. Determine, with explanation, the dimension of

(a) $\mathbb{C}$ as a vector space over $\mathbb{C}$;

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$;

(c) $\mathbb{C}^3$ as a vector space over $\mathbb{C}$;

(d) $\mathbb{C}^3$ as a vector space over $\mathbb{R}$.

Solution. First we notice that size of a list of basis vectors is constant despite which set of basis vectors we choose to represent a given vector space. Therefore, we can select any basis for a vector space and determine its dimension. The question asks for explanation, as a rigorous proof would likely call for more abstract algebra.

(a) $\dim_{\mathbb{C}}(\mathbb{C}) = 1$; can be written as $c \in \mathbb{C}$

(b) $\dim_{\mathbb{R}}(\mathbb{C}) = 2$; can be written as $a + bi; a, b \in \mathbb{R}$

(c) $\dim_{\mathbb{C}}(\mathbb{C}^3) = 3$; can be written as $\{c_1, c_2, c_3\}$; $c_i \in \mathbb{C}$

(d) $\dim_{\mathbb{R}}(\mathbb{C}^3) = 6$; can be written as $\{(a_1 + b_1 1), (a_2 + b_2 1), (a_3 + b_3 1)\}$;
same as $\{a_1, b_1, a_2, b_2, a_3, b_3\}$; $a_i, b_j \in \mathbb{R}$

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