Daniel Suryakusuma

SID: 24756460

Math 110, Spring 2019.

Homework 11, due Apr 21.

Prob 1. Let $T \in \mathcal{L}(V, W)$. Prove

(a) T is injective if and only if T^* is surjective;

(b) T^* is injective if and only if T is surjective.

Solution. In this solution we liberally use Axler 6.51, $U = [U^\perp]^\perp$ without further citation. Additionally we assume we know a linear transformation is injective if and only if its null space is zero.

(a) T is injective if and only if T^* is surjective;

First we prove the forward ($\implies$) direction. If T is injective, $\ker T = 0$. But Axler 7.7(c) gives $\ker T = [\text{Im } T^*]^\perp$. So $\ker T = 0 = [\text{Im } T^*]^\perp \implies \text{Im } T^* = 0^\perp = V$ by Axler 6.46(b), precisely the requirement for T^* surjective.

Now for the backwards ($\impliedby$) direction, suppose we have T^* surjective. Then $\text{Im } T^* = V$. But Axler 7.7(b) gives $\text{Im } T^* = [\ker T]^\perp$, so $V = [\ker T]^\perp$, so $\ker T = V^\perp = 0$ by Axler 6.46(c), and we thus have T injective as its kernel is 0.

(b) T^* is injective if and only if T is surjective.

First we prove the forward ($\implies$) direction. Suppose T^* is injective, then $\ker T^* = 0$. By Axler 7.7(a), $\ker T^* = [\text{Im } T]^\perp$. So $0 = [\text{Im } T]^\perp \implies \text{Im } T = 0^\perp = W$ by Axler 6.46(b), and T surjective as required.

Now for the backwards ($\impliedby$) direction, suppose T is surjective. Then $\text{Im } T = W$. By Axler 7.7(d), $\text{Im } T = [\ker T^*]^\perp$. So $\ker T^* = W^\perp = 0$ by Axler 6.46(c), and T injective (as its kernel is zero) as required.

□

Note to self: Recall the "adjoint of T," $T^* \in \mathcal{L}(W, V)$, is defined with the condition that for all $v \in V, w \in W$,

$$\langle Tv, w \rangle = \langle v, T^* w \rangle.$$

Prob 2. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Solution. Given S, T Hermitian, then $S = S^*$ and $T = T^*$. Moreover, $\forall v, v' \in V$, $\langle Tv, w\rangle = \langle v, Tw\rangle$.
($\implies$) Suppose ST is Hermitian. Then $ST = [ST]^*$, so we have:

$$\begin{aligned} ST = [ST]^* &= T^*S^* \text{ , by Axler 7.6(e), properties of adjoint} \\ &= T^*[S] \text{ hypothesis, } S = S^* \\ &= TS \text{ hypothesis, } T = T^* \end{aligned}$$

So $ST = TS$, as desired.

($\impliedby$) Suppose $ST = TS$. Performing substitutions, we have

$$\begin{aligned} ST = (S)(T) &= S^*T^* \text{ hypothesis S,T Hermitian; substituting } S = S^*, T = T^* \\ &= [TS]^* \text{ Axler 7.6(e), } [TS]^* = S^*T^* \\ &= [ST]^* \text{ hypothesis, } ST = TS \\ &\implies TS = ST = [ST]^* = [TS]^* \end{aligned}$$

In other words TS and ST are BOTH self-adjoint (with their adjoints being equal), which of course proves ST is Hermitian as desired.

□

Prob 3. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Solution. We are given $P^2 = P$.

($\Longrightarrow$) Suppose there exists subspace $U \subset V$ with $P_U = P = P^2$.

With respect to our given U, we can write $v = u + w$ and $v' = u' + w'$, where $u, u' \in U$ and $w, w' \in U^\perp$.

Then $P(v) = P^2(v) = P_U(v) = u$, and similarly $P(v') = u'$ for all $v, v' \in V$. Consider:

$$\begin{aligned}\langle Pv, v' \rangle &= \langle u, u' + w' \rangle, \quad (P(v) = u) \\ &= \langle u, u' \rangle + \langle u, w' \rangle, \quad \text{(properties of inner product)} \\ &= \langle u, u' \rangle + 0, \quad (u \in U, w' \in U^\perp \implies \langle u, w' \rangle = 0) \\ &= \langle u, u' \rangle + \langle w, u' \rangle, \quad (u' \in U, w \in U^\perp \implies \langle w, u' \rangle = 0) \\ &= \langle u + w, u' \rangle, \quad \text{(properties of inner product)} \\ &= \langle v, Pv' \rangle, \quad (P(v') = u')\end{aligned}$$

So for all $v, v' \in V$, we have $\langle Pv, v' \rangle = \langle v, Pv' \rangle$, precisely the requirement for P self-adjoint.

($\Longleftarrow$) Suppose P is self-adjoint (Hermitian). Then $P = P^* = P^2$. Surely, finite-dimensional subspaces $U \subset V$ exist for our given V, so let this U be given and fixed. Consider the set of such P that satisfy our given initial conditions:

$$S := \{P | P = P^2, P = P^*\}.$$

We need to show there exists some U, with $P_U \in S$. That is, for some U, we check if $P = P_U$ satisfies $P = P^2$ and $P = P^*$. If so, such U exists that satisfies our requirements, and we are done. Recall that $P_U(u) = u$, so $P_U[u] = P_U[P_U(u)] = [P_U]^2(u)$. It only remains to show $P_U = P_U^*$, which is to show for all $v, v' \in V$ that $\langle P_U v, v' \rangle = \langle v, P_U v' \rangle$. But we have already done this above in the forward direction (letting $P = P_U$). So $P_U \in S$, and such U exists with the specifications $P = P^2$ and $P = P_U$.

$\square$

Prob 4. Let $n \in \mathbb{N}$ be fixed. Consider the real space $V := \text{span}(1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx)$ with inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ is *anti-Hermitian*, i.e., satisfies $D^* = -D$.

Solution. First establish an orthonormal basis for V. We claim that for the above inner product, the list L is an orthonormal basis for V (given finite n): $L := \{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}\}$

Span follows immediately by definition of V. Linear independence can be seen from distinct elements of L being orthogonal. Let $j \neq k$. Note the following integration formulas can be found by integration by parts / tabular method.

$$\int_{-\pi}^{\pi} [\cos kt][\sin kt]\ dt = -\frac{[\cos kt]^2}{2k}\Big|_{-\pi}^{\pi} = 0$$
$$\int_{-\pi}^{\pi} [\cos kt][\cos jt]\ dt = \frac{j\sin(j-k)t + k\sin(j-k)t + j\sin(j+k)t - k\sin(j+k)t}{2(j+k)(j-k)}\Big|_{-\pi}^{\pi} = 0$$
$$\int_{-\pi}^{\pi} [\cos kt][\sin jt]\ dt = -\frac{j\cos(j-k)t + k\cos(j-k)t + j\cos(j+k)t - k\cos(j+k)t}{2(j+k)(j-k)}\Big|_{-\pi}^{\pi} = 0$$
$$\int_{-\pi}^{\pi} [\sin kt][\sin jt]\ dt = \frac{j\sin(j-k)t + k\sin(j-k)t - j\sin(j_k)t + k\sin(j+k)t}{2(j+k)(j-k)}\Big|_{-\pi}^{\pi} = 0$$

So each element in L is accounted for and thus is orthogonal to each distinct element. Because L is an orthogonal list, it is linearly independent. It remains to show their norms are 1.

$\langle \frac{\cos kt}{\sqrt{\pi}}, \frac{\cos kt}{\sqrt{\pi}} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 kt\ dt = \frac{1}{\pi}\left[\frac{2kt+\sin(2kt)}{4k}\right]_{-\pi}^{\pi} = 1$

$\langle \frac{\sin kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 kt\ dt = \frac{1}{\pi}\frac{2kt-\sin(2kt)}{4k}\Big|_{-\pi}^{\pi} = 1$

$\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \rangle = 1$

Thus L gives and orthonormal basis of V, as desired. Let us write L as (where $e_k := \frac{\cos kx}{\sqrt{\pi}}$ and $f_k := \frac{\sin kx}{\sqrt{\pi}}$): $\{\frac{1}{\sqrt{2\pi}}, e_1, \dots, e_n, f_1, \dots, f_n\}$.

Let $v, w \in V$. Then we have:

$v = \langle v, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}[\langle v, e_k\rangle e_k + \langle v, f_k\rangle f_k]$ and

$w = \langle w, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}[\langle w, e_k\rangle e_k + \langle w, f_k\rangle f_k]$

Then consider:

$$\begin{aligned}
\langle Dv, w\rangle &= \left\langle D\left[\langle v, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}[\langle v, e_k\rangle e_k + \langle v, f_k\rangle f_k]\right], \left[\langle w, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}[\langle w, e_k\rangle e_k + \langle w, f_k\rangle f_k]\right]\right\rangle \\
&= \left\langle 0 + \sum_{k=1}^{n}\left[-k\langle v, e_k\rangle f_k + k\langle v, f_k\rangle e_k\right], \left[\langle w, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}[\langle w, e_k\rangle e_k + \langle w, f_k\rangle f_k]\right]\right\rangle \\
&= \left\langle \sum_{k=1}^{n}\left[-k\langle v, e_k\rangle f_k + k\langle v, f_k\rangle e_k\right], \left[\sum_{k=1}^{n}[\langle w, e_k\rangle e_k + \langle w, f_k\rangle f_k]\right]\right\rangle + \left\langle Dv, \langle w, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}}\right\rangle \\
&= \sum_{k=1}^{n}\left[-k\langle v, e_k\rangle\langle w, f_k\rangle + k\langle v, f_k\rangle\langle w, e_k\rangle\right] \\
&= \left\langle \langle v, \frac{1}{\sqrt{2\pi}}\rangle\frac{1}{\sqrt{2\pi}}, -Dw\right\rangle + \left\langle \sum_{k=1}^{n}\left[\langle v, e_k\rangle e_k + \langle v, f_k\rangle f_k\right], -Dw\right\rangle = \langle v, -Dw\rangle = \langle v, D^*w\rangle
\end{aligned}$$

$\square$

Prob 5. Let T be a normal operator on V. Evaluate $\|T(v-w)\|$ given that

$$Tv = 2v, \qquad Tw = 3w, \qquad \|v\| = \|w\| = 1.$$

Solution. Recall Axler 7.22 gives "Eigenvectors of normal T corresponding to distinct eigenvalues are orthogonal." Then v is the eigenvector of T corresponding to 2, and w is that of T corresponding to 3. Thus $\langle v, w\rangle = 0$.

$$\begin{aligned}
\text{Consider:} \quad ||T(v-w)|| &= ||T(v) - T(w)|| \\
&= ||2v - 3w|| \quad (Tv = 2v, Tw = 3w) \\
&= \sqrt{\langle 2v - 3w, 2v - 3w\rangle} \\
&= \sqrt{\langle 2v, 2v - 3w\rangle - \langle 3w, 2v - 3w\rangle} \\
&= \sqrt{\langle 2v, 2v\rangle - \langle 2v, 3w\rangle - \langle 3w, 2v\rangle + \langle 3w, 3w\rangle} \\
&= \sqrt{\langle 2v, 2v\rangle - 0 - 0 + \langle 3w, 3w\rangle} \quad \text{(Axler 7.22, } v, w \text{ orthogonal)} \\
&= \sqrt{||2v||^2 + ||3w||^2} \\
&= \sqrt{4||v||^2 + 9||w||^2} \\
&= \sqrt{4+9} \quad (||v|| = ||w|| = 1) \\
&= \sqrt{13}
\end{aligned}$$

□

Note this is much shorter if we allow (Axler's) pythagorean theorem.

Prob 6. Suppose T is normal. Prove that, for any $\lambda \in \mathbb{F}$ and any $k \in \mathbb{N}$,

$$\text{Null}\,(T-\lambda I)^k = \text{Null}\,(T-\lambda I).$$

Solution. Fix given $\lambda \in \mathbb{F}$ and $k \in \mathbb{N}$. Given T is normal, $S := (T-\lambda I)$ is also normal (Lemma 1). Consider the set U of $k \in \mathbb{N}$ for which $\ker(T-\lambda I)^k = \ker(T-\lambda I)$. First note that it need not be true that $\ker I = \ker[T-\lambda I]$, so $0 \notin U$. But $\ker(T-\lambda I)^1 = \ker(T-\lambda I)$, so $1 \in U$. Now consider $k \geq 2$.

First we show $\ker S^k \subset \ker S$. Take $v \in \ker S^k$. Then $S^k(v) = 0$. Consider

$$\begin{aligned}
\langle S^*S^{k-1}v, S^*S^{k-1}v\rangle &= \langle SS^*S^{k-1}v, S^{k-1}v\rangle,\ \text{definition of adjoint}(\langle Tv, v'\rangle = \langle v, T^*v'\rangle)\\
&= \langle S^*SS^{k-1}v, S^{k-1}v\rangle,\ S \text{ normal (lemma 1), } S \text{ and } S^* \text{ commute}\\
&= \langle S^*[S^k v], S^{k-1}v\rangle\\
&= \langle 0(v), S^{k-1}v\rangle,\ (S^k(v) = 0 \implies S^*[S^k(v)] = 0)\\
&= 0 \implies 0 = S^*S^{k-1}(v) \text{ by properties of inner product}\\
\text{consider: } 0 &= \langle 0(v), S^{k-2}(v)\rangle \quad (\langle 0(v), v\rangle = 0 \forall v \in V)\\
&= \langle S^*S^{k-1}(v), S^{k-2}(v)\rangle\ \ , (S^*S^{k-1}(v) = 0,\ \text{above})\\
&= \langle S^{k-1}(v), S^{k-1}(v)\rangle \implies S^{k-1}(v) = 0 \implies v \in \ker S^{k-1}
\end{aligned}$$

Because $k \in \mathbb{N}$, by definition of natural numbers, our recursion above $k \implies k-1$ gives for some fixed k and for all $v \in V$, that $v \in \ker S^k$ implies $v \in \ker S$. Thus we have $\ker S^k \subset \ker S$, as desired.

Next we (easily) show $\ker S^k \supset \ker S$. Take $v \in \ker S$. Then by definition of kernel, $S(v) = 0$. Consider $S^k(v) = S^{k-1}[S(v)] = S^{k-1}[0] = 0$. So $S^k(v) = 0$, and $v \in \ker S^k$. Hence $\ker S^k \supset \ker S$. We have shown $\ker S^k \subset \ker S$ and $\ker S^k \supset \ker S$, so $\ker S^k = \ker S$ as desired.

□

Lemma 1. Given T is normal, then for any $\lambda \in \mathbb{F}$, $S := (T-\lambda I)$ is normal.

Proof. If $S = (T-\lambda I)$ commutes with its adjoint, then S is normal. T is normal so $TT^* = T^*T$. Consider

$$\begin{aligned}
SS^* &= (T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \overline{\lambda} I)\\
&= TT^* - \lambda T^* - \overline{\lambda} T + \lambda\overline{\lambda} = T^*T - \lambda T^* - \overline{\lambda} T + \lambda\overline{\lambda}\\
&= (T-\lambda I)^*(T-\lambda I) = (T^* - \overline{\lambda} I)(T-\lambda I) = S^*S.
\end{aligned}$$

□