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Homework 10, due April 14.

Prob 1. Let e_1, ..., e_m be an orthonormal list of vectors. Prove that

$$v \in \operatorname{span}\{e_1, \dots, e_m\} \iff \|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j\rangle|^2.$$

Solution. We first prove the forward direction ($\implies$). Given $e_1, \dots, e_m$ is an orthonormal list of vectors, because they are linearly independent (Axler 6.26), we can uniquely write, for some scalars $c_j \in \mathbb{F}$,

$$v = \sum_{j=1}^{m} c_j e_j = c_1 e_1 + \cdots + c_m e_m. \tag{1}$$

Because the list $e_1, \dots, e_m$ is orthonormal, "taking the inner product of both sides of this equations with e_j gives $\langle v, e_j\rangle = a_j$" (as given by Axler's proof of 6.30). To see this explicitly, consider the following:

$$\begin{aligned}
\langle v, e_1\rangle &= \langle [c_1e_1 + \cdots + c_m e_m], e_1\rangle = c_1 + 0 + \cdots + 0 \\
\langle v, e_2\rangle &= \langle [c_1e_1 + \cdots + c_m e_m], e_2\rangle = 0 + c_2 + \cdots + 0 \\
&\vdots \\
\langle v, e_m\rangle &= \langle [c_1e_1 + \cdots + c_m e_m], e_m\rangle = 0 + \cdots + 0 + c_m
\end{aligned}$$

Recall that Axler 6.25 gives "If $e_1, \dots, e_m$ is an orthonormal list of vectors in V, then $\forall a_1, \dots, a_m \in \mathbb{F}$, $||a_1e_1 + \cdots + a_m e_m||^2 = |a_1|^2 + \cdots + |a_m|^2$." Then consider the following;

$$\begin{aligned}
||v||^2 &= ||c_1e_1 + \cdots + c_m e_m||^2 \\
&= |c_1|^2 + \cdots + |c_m|^2 \\
&= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 \\
&= \sum_{j=1}^{m} |\langle v, e_j\rangle|^2
\end{aligned}$$

Now we prove the backwards ($\impliedby$) direction. Given $||v||^2 = \sum_{j=1}^{m} |\langle v, e_j\rangle|^2$, we wish to show that $v \in \operatorname{span}\{e_1, \dots, e_m\}$. Suppose $v \notin \operatorname{span}\{e_1, \dots, e_m\}$. Then $v \neq \sum_{j=1}^{m} c_j e_m$ for all $c_j \in \mathbb{F}$, so

$$(v \neq c_1e_1 + \cdots + c_m e_m) \implies (||v||^2 \neq ||c_1e_1 + \cdots + c_m e_m||^2) \implies (||v||^2 \neq \sum_{j=1}^{m} |\langle v, e_j\rangle|^2),$$

which is a direct contradiction to our hypothesis that $||v||^2 = \sum_{j=1}^{m} |\langle v, e_j\rangle|^2$. So it must be true that $v \in \operatorname{span}\{e_1, \dots, e_m\}$.

$\square$

Prob 2. Consider the space $\mathcal{P}_3(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, x^2, x^3$.

Solution. Recall the Gram-Schmidt algorithm for "orthonormalizing" any given basis $v_1, \ldots, v_m$ is outlined recursively (Axler 6.31) as follows:

Let $e_1 = \frac{v_1}{|v_1|}$. Then recursively for incrementing $j = 2, \ldots, m$, let: $e_j = \frac{v_j - \sum_{i=1}^{j-1}\langle v_j, e_i\rangle e_i}{\left|v_j - \sum_{i=1}^{j-1}\langle v_j, e_i\rangle e_i\right|}$.

With our given inner product space over $\mathcal{P}_3(\mathbb{R})$ and ordered basis $\{1, x, x^2, x^3\}$, we simply follow the procedure (our orthonormal basis is e_1, e_2, e_3, e_4 as follows):

$$e_1 = \frac{1}{|1|} = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{\int_{-1}^{1} 1dx}} = \frac{1}{\sqrt{2}}$$

$$e_2 = \frac{x - \langle x, \frac{1}{\sqrt{2}}\rangle\frac{1}{\sqrt{2}}}{|x - \langle x, \frac{1}{\sqrt{2}}\rangle\frac{1}{\sqrt{2}}|} = \frac{x}{|x|} = \frac{x}{\sqrt{\langle x, x\rangle}} = \frac{x}{\sqrt{2/3}}$$

$$e_3 = \frac{x^2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}}}{|x^2 - \frac{1}{3}|} = \frac{x^2 - \frac{1}{3}}{\sqrt{8/45}} = \frac{[3x^2 - 1]\sqrt{10}}{4}$$

$$e_4 = \frac{x^3 - 0e_1 - \langle x^3, e_2\rangle e_2 - 0e_3}{\left|x^3 - 0e_1 - \langle x^3, e_2\rangle e_2 - 0e_3\right|} = \frac{x^3 - \frac{\sqrt{6}}{5}\frac{x}{\sqrt{2/3}}}{|x^3 - \frac{\sqrt{6}}{5}\frac{x}{\sqrt{2/3}}|} = \frac{x^3 - \frac{3x}{5}}{|x^3 - \frac{3x}{5}|} = \sqrt{\frac{175}{8}}\left[x^3 - \frac{3x}{5}\right] = \frac{7}{2\sqrt{2}}[5x^3 - 3x]$$

Explicit calculations for the inner products required above are as follows:

$$\langle x, 1/\sqrt{2}\rangle = \int_{-1}^{1}\frac{1}{\sqrt{2}}xdx = \frac{1}{2\sqrt{2}}x^2\Big|_{-1}^{1} = 0$$

$$\langle x, x\rangle = \int_{-1}^{1} x^2dx = \frac{1}{3}x^3\Big|_{-1}^{1} = 2/3$$

$$\langle x^2, e_1\rangle = \int_{-1}^{1}\frac{1}{\sqrt{2}}x^2dx = \frac{1}{3\sqrt{2}}x^3\Big|_{-1}^{1} = \sqrt{2}/3$$

$$\langle x^2, e_2\rangle = \int_{-1}^{1}(\frac{x}{\sqrt{2/3}})x^2dx = \sqrt{3/2}\frac{1}{4}x^4\Big|_{-1}^{1} = 0$$

$$\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = \int_{-1}^{1}[x^2 - 1/3]^2dx = \left[\frac{x^5}{5} - 2x^3 + \frac{x}{9}\right]_{-1}^{1} = 8/45$$

$$\langle x^3, e_1\rangle = \int_{-1}^{1}(x^3)[\frac{1}{\sqrt{2}}]dx = 0$$

$$\langle x^3, e_2\rangle = \int_{-1}^{1}(x^3)[\frac{\sqrt{6}x}{2}]dx = \frac{\sqrt{6}x^5}{10}\Big|_{-1}^{1} = \frac{\sqrt{6}}{5}$$

$$\langle x^3, e_3\rangle = \int_{-1}^{1}(x^3)[\frac{[3x^2 - 1]\sqrt{10}}{4}]dx = \frac{\sqrt{10}}{4}\left[\frac{x^6}{2} - \frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\langle x^3 - \frac{3x}{5}, x^3 - \frac{3x}{5}\rangle = \int_{-1}^{1}\left[x^3 - \frac{3x}{5}\right]^2 dx = \int_{-1}^{1}[x^6 + \frac{9x^2}{25} - \frac{6x^4}{5}]dx = \left[\frac{x^7}{7} + \frac{3x^3}{25} - \frac{6x^5}{25}\right]_{-1}^{1} = \frac{2}{7} + \frac{6}{25} - \frac{12}{25} = \frac{8}{175}$$

□

Prob 3. Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |1 + 4x - p(x)|^2 dx$$

is as small as possible.

Solution. Consider an inner product defined for $f, g \in \mathcal{P}_3(\mathbb{R})$ as $\langle f, g\rangle := \int_0^1 |f||g|\ dx$. We first verify this satisfies the requirements for an inner product. Due to properties of polynomials, for $\lambda \in \mathbb{R}, u, v, w \in \mathcal{P}(\mathbb{R})$, we have additivity ($\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$), homogeneity ($\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$), and conjugate symmetry (although here $\mathbb{F} = \mathbb{R}$). And because we multiply two absolute valued polynomials $|f|, |g|$ then our integral over $[0, 1]$ surely is positive with $\langle v, v\rangle = 0 \iff v = 0$. As all the required properties of an inner product are satisfied, our inner product is well defined.

Because $p \in \mathcal{P}_3(\mathbb{R})$, we can write for some scalars $a_0, a_1, a_2, a_3 \in \mathbb{R}$, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. However, $p(0) = 0 \implies a_0 = 0$ and $p'(0) = 0 \implies a_1 = 0$. So we can write

$$p(x) = a_2x^2 + a_3x^3.$$

Let U be the subspace of $\mathcal{P}_3(\mathbb{R})$ spanned by $p(x)$ above for $a_2, a_3 \in \mathbb{R}$. Due to the absolute value in our integral, our main problem of minimizing the given integral is equivalent to finding some $u \in U$ that minimizes $|(1+4x) - u|$. In other words, finding $u \in U$ that minimizes the "distance" from $(1 + 4x)$. Proven in Axler, this occurs precisely for $u \in U$ with

$$u = P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m,$$

given $e_1, \ldots, e_m$ is an orthonormal basis of U.

We have written for any $u \in U$, $u = a_2x^2 + a_3x^3$. Take the canonical monomial basis, x^2, x^3 for U. Performing Gram-Schmidt on this basis, we get:

$$\begin{aligned}
e_1 &:= \frac{x^2}{|x^2|} = \frac{x^2}{\sqrt{\langle x^2, x^2\rangle}} = \frac{x^2}{\sqrt{\int_0^1 |x^2||x^2|\ dx}} = \frac{x^2}{\sqrt{\frac{1}{5}}} = \sqrt{5}x^2 \\
e_2 &:= \frac{x^3 - \langle x^3, e_1\rangle e_1}{|x^3 - \langle x^3, e_1\rangle e_1|} = \frac{x^3 - \frac{5x^2}{6}}{|x^3 - \frac{5x^2}{6}|} = \frac{x^3 - \frac{5x^2}{6}}{\sqrt{\langle x^3 - \frac{5x^2}{6}, x^3 - \frac{5x^2}{6}\rangle}} \\
&= \frac{x^3 - \frac{5x^2}{6}}{\sqrt{\int_0^1 |x^3 - \frac{5x^2}{6}||x^3 - \frac{5x^2}{6}|\ dx}} = \sqrt{7}[6x^3 - 5x^2]
\end{aligned}$$

Then we have, for $v := 1 + 4x$,

$$\begin{aligned}
u = P_U(v) &= \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m \\
&= \left\langle 1 + 4x, \sqrt{5}x^2\right\rangle \sqrt{5}x^2 + \left\langle 1 + 4x, \sqrt{7}[6x^3 - 5x^2]\right\rangle \sqrt{7}[6x^3 - 5x^2] \\
&= \left[\sqrt{5}x^2 \int_0^1 |1 + 4x||\sqrt{5}x^2|\ dx\right] + \left[\sqrt{7}[6x^3 - 5x^2] \int_0^1 |1 + 4x||\sqrt{7}[6x^3 - 5x^2]|\ dx\right] \\
&= \frac{20x^2}{3} + \frac{-77[6x^3 - 5x^2]}{30} \\
&= \frac{39x^2}{2} - \frac{77x^3}{5}
\end{aligned}$$

So $p = \frac{39x^2}{2} - \frac{77x^3}{5} \in \mathcal{P}_3(\mathbb{R})$ satisfies $p(0) = p'(0) = 0$ and minimizes our given integral expression.

□

Prob 4. Consider a complex vector space $V = \text{span}\,(1, \cos x, \sin x, \cos 2x, \sin 2x)$ with an inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

Let U be the subspace of odd functions in V. What is $U^{\perp}$? Find an orthonormal basis for both U and $U^{\perp}$.

Solution. We are given $V = \text{span}\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$. First verify these are linearly independent. Suppose we have for some $a_i \in \mathbb{C}$, that $a_0 + a_1 \cos x + a_2 \sin x + a_3 \cos 2x + a_4 \sin 2x = 0$. Recall from trigonometric identities that $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$. Then obviously $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is a linearly independent list spanning V (and thus a basis of V). Then any $v \in V$ can be written as a linear combination of our basis vectors. That is, $\forall v \in V$, for some $a_i \in \mathbb{C}$,

$$v = a_0 + a_1 \cos x + a_2 \sin x + a_3 \cos 2x + a_4 \sin 2x.$$

If U is the subspace of odd functions in V, from the definition of even function and odd function, $1(-x) = 1 = 1(x), \cos(-x) = \cos(x), \cos(-2x) = \cos(2x)$, so we have that $1, \cos x, \cos 2x \notin U$. Likewise $\sin(-x) = -\sin(x), \sin(-2x) = -\sin(2x)$, so $\sin x, \sin 2x \in U$. Because our list is a basis for V, and $U \subset V$, we have:

$$U = \{v \in V \mid a_0 = a_1 = a_3 = 0\}$$

It is easily verified from dimension 2 and linear independence that $\{\sin x, \sin 2x\}$ is a basis for U. But of course, any basis can be "orthonormalized" via Gram-Schmidt. We want an orthonormal basis of U, say $\{e_1, e_2\}$.

$$e_1 := \frac{\sin x}{|\sin x|} = \frac{\sin x}{\sqrt{\langle \sin x, \sin x\rangle}} = \frac{\sin x}{\sqrt{\int_{-\pi}^{\pi} \sin x \overline{\sin x}\ dx}} = \frac{\sin x}{\sqrt{\pi}}$$

$$e_2 := \frac{\sin 2x - \langle \sin 2x, \frac{\sin x}{\sqrt{\pi}}\rangle \frac{\sin x}{\sqrt{\pi}}}{\left|\sin 2x - \langle \sin 2x, \frac{\sin x}{\sqrt{\pi}}\rangle \frac{\sin x}{\sqrt{\pi}}\right|} = \frac{\sin 2x - \int_{-\pi}^{\pi} \frac{2\sin^2 x \cos x}{\sqrt{\pi}}\ dx \frac{\sin x}{\sqrt{\pi}}}{\left|\sin 2x - \int_{-\pi}^{\pi} \frac{2\sin^2 x \cos x}{\sqrt{\pi}}\ dx \frac{\sin x}{\sqrt{\pi}}\right|}$$
$$= \frac{\sin 2x - 0}{\left|\sin 2x - 0\right|} = \frac{\sin 2x}{\sqrt{\langle \sin 2x, \sin 2x\rangle}} = \frac{\sin 2x}{\sqrt{\pi}}$$

Axler gives $V = U \oplus U^{\perp}$. So $\dim V = \dim U + \dim U^{\perp}$. We established above that $1, \cos x, \cos 2x \notin U$, but this list is linearly independent (and of correct size, 3). So $1, \cos x, \cos 2x$ is a basis of $U^{\perp}$.

Let e_3, e_4, e_5 be an orthonormal basis (given by Gram-Schmidt) for $U^{\perp}$.

$$e_3 := \frac{1}{|1|} = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{\int_{-\pi}^{\pi} 1\ dx}} = \frac{1}{\sqrt{2\pi}}$$

$$e_4 := \frac{\cos x - \langle \cos x, e_3\rangle e_3}{\left|\cos x - \langle \cos x, e_3\rangle e_3\right|} = \frac{\cos x}{\sqrt{\pi}}$$

$$e_5 := \frac{\cos 2x - \langle \cos 2x, e_3\rangle e_3 - \langle \cos 2x, e_4\rangle e_4}{\left|\cos 2x - \langle \cos 2x, e_3\rangle e_3 - \langle \cos 2x, e_4\rangle e_4\right|} = \frac{\cos 2x}{\sqrt{\pi}}$$

So we have $\{\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}\}$ is an orthonormal basis of U, and $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}\}$ is an orthonormal basis of $U^{\perp}$, where $U^{\perp}$ is the "orthogonal complement" of U, the subset of V consisting of all vectors orthogonal to subspace U. □

Prob 5. Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if

$$P_U T P_U = T P_U.$$

Solution. First we prove the forward ($\implies$) direction. For all $v \in V$, we can uniquely write $v = u + w$, where $u \in U, w \in U^{\perp}$ (given by Axler's definition of orthogonal projection). Suppose U is invariant under T. Then by definition of T-invariant, $\forall u \in U, T(u) \in U$. But by definition of P_U, Axler gives: $P_U(v) = u$ for all $v \in V$.

That is,

$$P_U(v) = u \implies \forall v \in V, [P_U T P_U](v) = P_U T(u) = T(u) = T P_U(v).$$

Because this holds for all $v \in V$, this gives our desired statement $P_U T P_U = T P_U$.

Now we prove the backwards ($\impliedby$) direction. Suppose $P_U T P_U = T P_U$. If we show $T(u) \in U$, we have U is T-invariant as desired. Let $u \in U$. For any finite dimensional $U \subset V$, it is trivially so that $P_U(u) = u$, by definition of orthogonal projection.

$$\begin{aligned} P_U[T P_U(u)] &= P_U[T(u)] \text{ because } P_U(u) = u \\ &= T P_U(u) \text{ hypothesis, } P_U T P_U = T P_U \\ &= T(u) \text{ because } P_U(u) = u \end{aligned}$$

Then it must be so that $P_U[T(u)] = T(u)$, which implies by definition of orthogonal projection that $T(u) \in U$ for all $u \in U$. This precisely gives U is T-invariant, as desired.

$\square$