Daniel Suryakusuma

SID: 24756460

Math 110, Spring 2019.

Homework 1, due Feb 1.

Prob 1. Suppose $a \in \mathbb{F}$ (field), v, $w \in V$ (vector space over $\mathbb{F}$), and $av = aw$. Prove that $a = 0$ or $v = w$.

Solution. If $a = 0$ then $av = 0$ and $aw = 0$, so $av = 0 = aw$ and we are done. Suppose $a \neq 0$. We are given $av = aw$, so we rewrite $a(v - w) = 0$. We have $a \neq 0$ so $a \in \mathbb{F}$ implies we can left-multiply by "a-inverse" a^{-1} to get

$$a^{-1}a(v - w) = a^{-1}0$$

$$1(v - w) = 0$$

$$v = w$$

$\square$

Prob 2. Let $n \in \mathbb{N}$. Is $\mathbb{Q}^n$ a vector space over $\mathbb{Z}$? Over $\mathbb{Q}$? Over $\mathbb{R}$? Explain.

Solution. Define $\mathbb{Q}^n$ to be the set of n-tuples with elements from $\mathbb{Q}$. First we note two of the three candidate fields in the problem are vector spaces on their self-respective fields (that is, $\mathbb{Z}$ is **NOT** a vector space over $\mathbb{Z}$ (and is not a field), but $\mathbb{Q}$ is a vector space over $\mathbb{Q}$, and $\mathbb{R}$ is a vector space over $\mathbb{R}$). That is, using the result from **Problem 4**, we verify that $\mathbb{Q}$ and $\mathbb{R}$ are vector spaces over themselves, and thus we reduce Probem 2 to a problem of subspaces. To prove $\mathbb{Q}^n$ is a vector space over $\mathbb{F}$, we can check if $\mathbb{Q}^n$ is a subspace of the vector space formed by $\mathbb{F}$.

($\mathbb{Q}^n$ as vector space over $\mathbb{Z}$)
Now consider the first part of the problem, $\mathbb{Z}$ as a candidate field over which $\mathbb{Q}^n$ might be a vector space. Recall $\mathbb{Z}$ is not a vector space over itself. $\mathbb{Z}$ is not a field. Professor Holtz answered the question ("can we have a vector field over something that is not a field?") in lecture on Jan 28, stating this is impossible. If we do not trust her words and need verification, then we simply look at the fact that $2 \in \mathbb{Z}$ has no multiplicative inverse in $\mathbb{Z}$, failing one of the required axioms of a field. By definition of a vector space needing to be over a field, $\mathbb{Q}^n$ **cannot be a vector space over $\mathbb{Z}$.**

For the next two fields, $\mathbb{Q}^n$ as a vector field inherits the field addition and (scalar) multiplication. We know that $\mathbb{R}$ and $\mathbb{Q}$ as fields have commutative and associative binary operations of multiplication and addition, and also that each is a vector space over itself. Thus all that remains to prove is that $\mathbb{Q}^n$ is a subspace of $\mathbb{Q}$ over $\mathbb{Q}$ and a subspace of $\mathbb{R}$ over $\mathbb{R}$.

($\mathbb{Q}^n$ as vector space over $\mathbb{Q}$)
We consider if $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$. We reason that element-wise operations addition and multiplication of elements of $\mathbb{Q}^n$ are closed operations by construction of $\mathbb{Q}^n$'s elements from $\mathbb{Q}$. We need not introduce example variables explicitly because if we suppose the statement is false, we would claim that an element of n-tuple $\mathbb{Q}^n$ exists that does not also live in $\mathbb{Q}$ and we reach a contradiction. Therefore $\mathbb{Q}^n$ **is a vector subspace of $\mathbb{Q}$ and is thus a vector space over $\mathbb{Q}$.**

($\mathbb{Q}^n$ as a vector space over $\mathbb{R}$)
Now we consider if $\mathbb{Q}^n$ is a vector subspace of $\mathbb{R}$. Let $a = \pi \in \mathbb{Q}^n$ and let $v \in \mathbb{Q}^n$. Notice that $av \notin \mathbb{Q}^n$, so $\mathbb{Q}^n$ over $\mathbb{R}$ is not closed under scalar multiplication. Therefore $\mathbb{Q}^n$ **is not a vector space over $\mathbb{R}$.**

□

Prob 3. Suppose that $\{0, 1, x\}$ is a field with exactly three elements. What do the addition and multiplication tables *have to be* in that case? Based on the addition and multiplication tables you get, check this is indeed a field. What is the natural way to think of this field (and of x)?

Solution. We let $\{0, 1, x\}$ be a field with exactly 3 elements. Recall that the axioms of a field are (for addition and multiplication) commutativity, associativity, existences of identity/natural elements, and existence of inverses, as well as distributivity of multiplication over addition. We design the following addition and multiplication tables such that these axioms are met. Eliminating the possibilities of completed tables that fail an axiom, we are left with one unique table for each operation that fulfills the required axioms.

Table 1: Addition Table

+	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

As later derived, $x = 2$, $y = 1$

Table 2: Multiplication Table

×	0	1	x
0	0	0	0
1	0	1	x
x	0	x	y

Constructing these tables, we have only one unique addition table (given some x) that does not directly fail axioms of fields. For multiplication tables, we have a similar case, but with $xx = y$ and because x is fixed and given, y is fixed. We wish to show that $y \in \{0, 1, x\}$.

We check associativity exhaustively, but this is quick to verify. By symmetry, we see that addition and multiplication is commutative. The additive identity (0) is in each row and column for addition, and the multiplicative identity (1) is in each row and column for multiplication. We then also verify distributivity of multiplication over addition similarly as we did associativity (exhaustively for ≤ 27 possibilities).

Constructing the addition/multiplication table, we satisfy all these axioms and only one remains for $\{0, 1, x\}$ to be a field: existence of inverses.

For addition, we have $1 + 1 = x$ $x + 1 = 0$ and $x + x = 1$, with all other entries trivial. For multiplication, we have $xx = y$, with all other entries trivial.

$1 + 1 = x$ implies $x = 2$, but $x + 1 = 0$, so we know $3 = 0$. Similarly, $x + x = 1$ so $2 + 2 = 1$, so $4 = 1$. Immediately modulo 3 comes to mind, and we can define $\{0, 1, x\}$ on a field modulo 3 to make all those statements true. Then, we define $x = 2$.

We see x must have a multiplicative inverse for $\{0, 1, x\}$ to be a field, which cannot be 0 or 1 because $0x = 0$ and $1x = x$. So, $xx = 1$ must be the case. This is in agreement with $x = 2$ as derived from the addition table.

Although mentioned above, we will explicitly state here: the natural way to think of this field is $\mathbb{Z}/3\mathbb{Z}$, which is modulo 3 arithmetic. And the natural way to think of x is 2 (mod 3). $\square$

Prob 4. Prove that any field $\mathbb{F}$ is also a vector space over itself, with the field addition used as vector addition, and the field multiplication used as scalar multiplication.

Solution. We wish to show that $\mathbb{F}$ is a vector space over itself, so that we can use this fact for the other problems in this homework 1.

We immediately notice that the axioms of a field (say $\mathbb{F}$) are very similar to the axioms of a vector space. Intuitively, the axioms are almost identical when $\mathbb{F}$ acts as a vector space on itself, so if we assume the axioms of a field to be true, and then satisfy all axioms of a vector space (field as vector space on itself), then we are done.

Let $a, b, c \in \mathbb{F}$. Assume the following axioms (5) of fields to be true:

- commutativity $(+)$ $(\times)$: $a + b = b + a$; $ab = ba$
- associativity $(+)$ $(\times)$: $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$
- $\exists$ identity for each $(+, 0)$ $(\times, 1)$, $0, 1 \in \mathbb{F}$
- $\exists$ inverse for each $(+, a + a^{-1} = 0)$, $(\times, b^{-1}b = 1)$, $a, b, a^{-1}, b^{-1} \in \mathbb{F}$
- distributivity of $(\times)$ over $(+)$: $a(b + c) = ab + ac$

Inherit these binary operations $(+)$ defined on vector addition and $(\times)$ defined on scalar multiplication. Let $a, b, c, a^{-1}, b^{-1} \in \mathbb{F}$ like above, and $x, y, z \in V := \mathbb{F}$. For axioms of a **vector space**, we immediately satisfy:

- commutativity $(+)$: $x + y = y + x$
- associativity $(+)$ $(\times)$: $(x + y) + z = x + (y + z)$; $(ab)x = a(bx)$
- $\exists$ identity for each $(+, 0)$ $(\times, 1)$, $0, 1 \in \mathbb{F}$: $0 + x = x$; $1x = x$
- $\exists$ additive inverse y $\forall$ x : $x + y = 0$
- distributivity of scalar over vectors : $a(x + y) = ax + ay$
- distributivity of vector on scalars : $(a + b)x = ax + bx$

These conditions are directly implied from the axioms of a field and the application of a field $\mathbb{F}$ as a vector space on itself, and these conditions are complete to qualify $\mathbb{F}$ defined as such to be a vector space on its own field.

$\square$

Prob 5. For which values of a is the set of all real-valued twice differential functions f on the interval $(0, \infty)$ such that $f''(2) = a$ (equipped with the usual addition of functions and multiplication by real scalars) a vector space over $\mathbb{R}$?

Solution. We are interested in finding values of a such that the set of all functions with $f^{(2)}(2) = a$ is a vector subspace of $C_2(\mathbb{R})$, the set of all twice-differentiable functions over $\mathbb{R}$. For this problem define $W := \{f : f''(2) = a\}$ as the set of all functions satisfying our condition.

To check if $f : f''(2) = a \subset C_2(\mathbb{R})$, and show that it is a vector subspace, we must show that the following axioms hold, for W subspace of V:

- i) W is nonempty
- ii) if $(x, y \in W,\ \alpha \in \mathbb{R}) \implies (\alpha x + y \in W)$

Or equivalently, these set of axioms, similarly for W subspace of V:

- a) $0 \in W$
- b) if $(x, y \in W) \implies (x + y \in W)$
- c) if $(x \in W, \alpha \in \mathbb{R}) \implies (\alpha x \in W)$

First we want to gain some intuition on what a should be such that $f''(2) = a$. Suppose we already have a subspace W, and consider axiom a) which should be satisfied. Then, we have the zero vector 0 which lives in W, meaning it satisfies the condition $f''(2) = a$. The second derivative of 0 at 2 is 0. Then surely, $f''(2) = a = 0$ and $a = 0$.

However, it remains to prove that $a = 0$ **necessarily makes** $W := \{f : f''(2) = 0\}$ form a vector subspace of $C_2(\mathbb{R})$. Note we have defined W around 0.

To do so, we look at axioms i) and ii). We have our condition $f''(2) = 0$ and know that $f = 0$ satisfies this condition. Because we have found an element in this set W, W is nonempty and i) is proven. It remains to show the linear properties axiom ii).

Let $\alpha \in \mathbb{R}$ and $\exists x, y \in W$ by hypothesis. By definition of W, x, y are functions that satisfy the condition $x''(2) = 0$ and $y''(2) = 0$. Let x and y be vectors that accomplish this. We know that $g := \alpha f$ is differentiable simply as $\alpha f'$. We extend this to see that function $\alpha x(\xi)$ is simply α times the function $x(\xi)$ and lives in W by definition of W.

Explicitly, that is: $(x(\xi) \in W) + (\alpha x''(2) = 0 = x''(2)) \implies (\alpha x(\xi) \in W)$. Similarly, adding two functions $x(\xi)$ and $y(\xi)$ both of whose second derivative at 2 equals 0 must combinedly have a derivative of 0 at 2, by properties of addition of 0. Explicitly, $x''(2) + y''(2) = 0 + 0 = 0$, so we have $x + y \in W$. Combining these two results, we have:
Given $(x, y \in W, \alpha \in \mathbb{R})$, then $(\alpha x(\xi) + y) \in W$, which was to be shown for axiom ii). Note that we have also proven axioms a) b) and c) and can form a conclusive statement that way as well.

As $a = 0$ forms a set $W := \{f : f''(2) = 0\}$ and upholds all axioms of a vector subspace of $C_2(\mathbb{R})$, we then conclude that $\{f : f''(2) = 0\}$ is indeed a vector subspace of $C_2(\mathbb{R})$.

$\square$