Math 104, Summer 2019

PSET #8 and 9 (due Monday, 8/5/2019)

Theorem 0.1. For the power series $\sum a_n x^n$, let

$$\beta := \limsup |a_n|^{1/n}, \qquad R := \frac{1}{\beta}.$$

Our convention is if $\beta = 0$, then set $R := +\infty$, and if $\beta = +\infty$, set $R := 0$. We are guaranteed the following:

- The power series converges for $|x| < R$;
- The power series diverges for $|x| > R$,

where we call R the radius of convergence of a power series.

Problem 23.1. For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(b) $\sum (\frac{x}{n})^n$

(e) $\sum (\frac{2^n}{n!}) x^n$

(h) $\sum (\frac{(-1)^n}{n^2 \cdot 4^n}) x^n$

Solution. (b) We proceed directly as given by Ross 23.1. Let $s_n := \sum \left(\frac{x}{n}\right)^n$ and $a_n := \left(\frac{1}{n}\right)^n$. Let

$$\beta := \limsup |a_n|^{1/n} = \limsup \left| \left(\frac{1}{n}\right)^n \right|^{1/n} = \limsup \left(\frac{1}{n}\right) = 0$$

and set $R := +\infty$ because $\beta = 0$. Hence we have that s_n converges $\boxed{\forall x \in \mathbb{R}}$.

(e) Now let $s_n := \sum \left(\frac{2^n}{n!}\right) x^n$, and $a_n := \frac{2^n}{n!}$. Let

$$\beta := \limsup |a_n|^{1/n} = \limsup \left| \frac{2^n}{n!} \right|^{1/n} = 0,$$

so we (again) set $R := +\infty$, and s_n converges $\boxed{\forall x \in \mathbb{R}}$.

(h) Now let $s_n := \sum \left[\frac{(-1)^n}{n^2 \cdot 4^n}\right] x^n$, with $a_n := \frac{(-1)^n}{n^2 \cdot 4^n}$. Let

$$\begin{aligned} \beta := \limsup |a_n|^{1/n} &= \limsup \left| \frac{(-1)^n}{n^2 \cdot 4^n} \right|^{1/n} = \limsup \left| \frac{-1}{n^{2/n} \cdot 4} \right| \\ &= \frac{1}{4} \limsup \left(\frac{1}{n}\right)^{2/n} = \frac{1}{4} \cdot 1 = \frac{1}{4}, \end{aligned}$$

so $R := \frac{1}{\beta} = 4$, and we have that s_n converges $\boxed{\forall x \in [-4, 4] \subset \mathbb{R}}$, because the $x = -41$ case is handled by the fact that $\sum \frac{1}{n^2}$ is known to converge ($p = 2$ series) and the $x = 4$ case is handled similarly (or more easily via AST). $\square$

Problem 23.2. Repeat Exercise 23.1 for the following:
(c) $\sum x^{n!}$
(d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

Solution. (c) Let $s_n := \sum a_n x^n = \sum x^{n!}$, where

$$a_n := \begin{cases} 1, & (\exists_{k\in\mathbb{N}} : n = k!) \\ 0, & \text{otherwise.} \end{cases}$$

We notice that there cannot be a finite count of 1's in a_n, so

$$\beta := \limsup |a_n|^{1/n} = \limsup |1|^{1/n} = 1,$$

and so we set $R := \frac{1}{\beta} = 1$. Now we check the bounds $x := \{-1, 1\}$. Notice that for $n \geq 2$, $n!$ is even (due to 2 being a factor), so $x^{n!}$ for $x = \{-1, 1\}$ gives the same result for each value, namely that $\sum 1$ does not converge (as $1^k = 1, \forall k$). Hence we conclude that s_n converges $\boxed{\forall x \in (-1, 1)}$.

(d) Now let $s_n := \sum \frac{3^n}{\sqrt{n}} x^{2n+1}$, with

$$a_n := \begin{cases} \frac{3^n}{\sqrt{n}}, & n \geq 3,\ n \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Now to check the bounds, notice that if $x \geq 3$, then s_n diverges (to see this quickly, simply consider the radius of convergence is 1 for a geometric series and $\sum \frac{1}{\sqrt{n}}$ diverges by the p-test, with $p = \frac{1}{2} \leq 1$). Now for $x := -1$, we established $2n + 1$ is always odd and hence a $(-1)^{2n+1}$ will not give an alternating series (this is tricky). We know also by $p = \frac{1}{2} \leq 1$ that $\sum \frac{-(3^n)}{\sqrt{n}}$ does not converge. Hence we conclude that s_n converges $\boxed{\forall x \in (-1, 1)}$. □

Problem 23.4. For $n = 0, 1, 2, 3, \ldots$, let $a_n := \left[\frac{4+2(-1)^n}{5}\right]^n$.

1. Find $\limsup (a_n)^{1/n}$, $\liminf (a_n)^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.
2. Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.
3. Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence of the series.

Solution. (1) First notice that Ross does not request $\limsup |a_n|^{1/n}$ but rather without the absolute value on a_n (this is because the inference requested in part (2) is purely about 'non'-power series convergence). Further notice that the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ in cases like these generally give different results for when n is even or odd. We investigate further and evaluate the requested expressions:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| = \liminf \left|\frac{(4 + 2(-1)^{n+1})^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(4 + 2(-1)^n)^n}\right| = \begin{cases} \liminf \left|\frac{6^{n+1}}{5\cdot 2^n}\right| = +\infty, & n \text{ odd} \\ \liminf \left|\frac{2^{n+1}}{5\cdot 6^n}\right| = 0, & n \text{ even} \end{cases} = \boxed{0}$$

$$\liminf (a_n)^{1/n} = \liminf \left(\frac{4 + 2(-1)^n}{5}\right) = \boxed{\frac{2}{5}}$$

$$\limsup (a_n)^{1/n} = \limsup \left(\frac{4 + 2(-1)^n}{5}\right) = \boxed{\frac{6}{5}}$$

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|\frac{(4 + 2(-1)^{n+1})^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(4 + 2(-1)^n)^n}\right| = \begin{cases} \limsup \left|\frac{6^{n+1}}{5\cdot 2^n}\right| = +\infty, & n \text{ odd} \\ \limsup \left|\frac{2^{n+1}}{5\cdot 6^n}\right| = 0, & n \text{ even} \end{cases} = \boxed{+\infty}.$$

(2) Now we consider: $\sum a_n$ and $\sum(-1)^n a_n$. Immediately, $\sum a_n$ **cannot converge** because a_n itself has $a_n > 1$ for infinitely many n (or equivalently, $a_n \not\to 0$). Now to consider $\sum(-1)^n a_n$, we first write out terms in the series:

$$\begin{aligned}\sum(-1)^n a_n &= (-1)^0\left[\cdots\right]^0 + (-1)^1\left[\frac{2}{5}\right]^1 + (-1)^2\left[\frac{6}{5}\right]^2 + (-1)^3\left[\frac{2}{5}\right]^3 + \cdots \\ &= 1 - \frac{2}{5} + \left(\frac{6}{5}\right) - \left(\frac{2}{5}\right)^3 + \cdots,\end{aligned}$$

and although this is an alternating series, it turns out all these considerations are unnecessary as $[(-1)^n a_n] \not\to 0$, and hence $\sum(-1)^n a_n$ **diverges**.

(3) Now considering the power series $\sum a_n x^n$ with coefficients a_n as defined above, we proceed just as we have 5 times before above: Let

$$\beta := \limsup |a_n|^{1/n} = \limsup \left|\frac{4+2(-1)^n}{5}\right| = \frac{6}{5},$$

so set $R := \frac{1}{\beta} = \frac{5}{6}$. Now we check the boundaries and by inspection conclude that our interval of convergence is all $\boxed{x \in \left(-\frac{-5}{6}, \frac{5}{6}\right)}$. Because this would be marked as begging the question, we make this painfully explicit:

$$x := -\frac{5}{6} \implies \sum a_n x^n = \sum \left|\frac{4+2(-1)^n}{5}\right|^n \left(\frac{-5}{6}\right)^n,$$

which involves a sum of an infinite terms of 1 (explicitly, at all even indices n), and hence cannot converge ($a_n x^n \not\to 0$). Setting $x := \frac{5}{6}$ gives the same result. If that isn't sufficient, consider:

$$x := \frac{5}{6} \implies \sum a_n x^n = \sum \left|\frac{4+2(-1)^n}{5}\right|^n \left(\frac{5}{6}\right)^n,$$

which again involves a sum of infinite terms of 1, again precisely at all even n. Finally, we conclude that the power series $\sum a_n x^n$ converges for all $\boxed{x \in \left(\frac{-5}{6}, \frac{5}{6}\right)}$.

□

Theorem 0.2. A sequence (f_n) of functions on a set $S \subset \mathbb{R}$ converges uniformly to a function f on S if and only if

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0.$$

Problem 24.2 For $x \in [0, \infty)$, let $f_n(x) := \frac{x}{n}$.
1. Find $f(x) := \lim f_n(x)$.
2. Determine whether $f_n \to f$ uniformly on $[0, 1]$.
3. Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution. (1) Fix any arbitrary $x \in [0, \infty)$, and notice $f(x) := \lim f_n(x) = \lim \frac{x}{n} = 0$, so we have **pointwise** (this distinction is important) convergence $f(x) = \lim f_n(x) = 0$.

(2) Now given by Ross, we say $f_n \to f$ uniformly on $[0, 1]$ if

$$\forall_{\epsilon>0} \exists_{N\in\mathbb{N}} : \forall_{x\in S, n>N} |f_n(x) - f(x)| < \epsilon.$$

We show this precisely. Fix $\epsilon > 0$ and any $x \in [0, 1]$. If $x = 0$, then $f_n(x) := \frac{0}{n} = f(x)$, and of course $|f_n(x)-f(x)| = |0 - 0| = 0$ (we need to handle this case separately as it is not given by the following). Now suppose $x \neq 0$. Simply take $N := \frac{x}{\epsilon}$. Then for $n > N$, we have:

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| < \left|\frac{x}{x/\epsilon}\right| = \epsilon,$$

which precisely gives uniform convergence of $f_n \to f$ on $[0, 1]$.

(3) Now we claim $f_n \not\to f$ on $[0, \infty)$. Given in the previous page, if $f_n \to f$ uniformly on $x \in [0, \infty)$, then we must have for all $x \in [0, \infty)$,

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in [0, \infty)\} = \lim_{n\to\infty} \sup \left\{ \left|\frac{x}{n}\right| : x \in [0, \infty) \right\} = 0.$$

As given by Ross, "If $f - f_n$ is differentiable, we may use calculus to find these suprema," even though we never defined a derivative. Hence we proclaim that $f_n(x) := \frac{x}{n}$ is differentiable (with respect to x), with

$$f_n'(x) := \frac{1}{n} > 0, \forall x \in [0, \infty),$$

where the strict inequality gives (unclaimed) intuition that f_n is strictly increasing unboundedly with respect to x and hence does not converge uniformly on $[0, \infty)$. We (do not admit we are stuck and) investigate further (by scrapping this completely and starting with the definition of uniform convergence).

Suppose (for contradiction) that $f_n \to f$ uniformly on $[0, \infty)$. Then (by definition of uniform continuity), for each $\epsilon > 0$ there exists a number N such that for all $x \in [0, \infty)$ and $n > N$, $|f_n(x) - f(x)| < \epsilon$. We show this uniformity fails: fix N arbitrarily (large) and consider $\epsilon := 1$. Take $x_0 = 2n + 2 \in [0, \infty]$ and notice

$$|f_{N+1}(x_0) - f(x_0)| = \left|\frac{x_0}{N+1}\right| = \left|\frac{2N+2}{N+1}\right| = 2 \not< 1,$$

and hence $f_n := \frac{x}{n}$ cannot be uniformly convergent to f.

□

Problem 24.3. Repeat exercise 24.2 for $f_n := \frac{1}{1+x^n}$. That is, for $x \in [0, \infty)$,
1. Find $f(x) := \lim f_n(x)$.
2. Determine whether $f_n \to f$ uniformly on $[0, 1]$.
3. Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution. (1) Considering case-work for $x \in [0, \infty)$, we claim the following point-wise convergence is self-evident:

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & 1 < x \end{cases}$$

(2) Take $x \in [0,1]$. If $x = 1$, then $f_n(x) = f(x)$ for all n (if this is not obvious, we write $\frac{1}{1+1^n} = \frac{1}{2}$). If it is not obvious that this equality implies $|f(1) - f_n(1)| < \epsilon$ for any $\epsilon < 0$, we expicitly state so here. Now suppose $x \neq 1$, so that $x \in [0,1)$ and $f(x) = 1$. Fix $\epsilon > 0$ and take $N := \max\{42, \left|\frac{\log \epsilon}{\log x}\right|\}$. Then for $x \in [0,1)$ and $n > N$,

$$\left|\frac{1}{1+x^n} - 1\right| = \left|\frac{1}{1+x^n} - \frac{1+x^n}{1+x^n}\right| = \left|\frac{x^n}{1+x^n}\right| < \frac{x^n}{1} < x^{\frac{\log \epsilon}{\log x}} = x^{\log_x \epsilon} = \epsilon,$$

as required to conclude $f_n \to f$ uniformly on $[0,1]$.

(3) Now take $x \in [0,\infty)$. Splitting into cases involves identical arguments to (2) above, so we notice that in (2) we already proved uniform continuity for $[0,1] \subset [0,\infty)$. It only remains to show that $f_n \to f$ uniformly on $(1,\infty)$. From part (1) earlier, for $x > 1$, we have $f(x) = \lim f_n(x) = 0$. Take $x \in (1,\infty)$ and $\epsilon > 0$. Let $N := \max\{42, \left|\log_x\left(\frac{1}{\epsilon}\right)\right|\}$. Then for all $n > N$ and $x \in (1,\infty)$, we have:

$$|f_n(x) - f(x)| = \left|\frac{1}{1+x^n} - 0\right| < \left|\frac{1}{x^{\log_x(1/\epsilon)}}\right| = \frac{1}{1/\epsilon} = \epsilon,$$

as required to show $f_n \to f$ uniformly on $(1,\infty)$. Because we have $f_n \to f$ uniformly on $[0,1]$ from part (2) above, we conclude $f_n \to f$ uniformly on $[0,\infty)$ which was to be shown. □

Problem 25.5. Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

Solution. Because (f_n) is a sequence of bounded functions on set S, for $i \in \mathbb{N}$, let m_i be a bound for f_i, so that $|f_i(x)| < m_i$ for all $x \in S$. For each $n \in \mathbb{N}$, define $M_n := \max\{m_i : 1 \leq i \leq n\}$. Notice that M_n is a bound for all $f_1, f_2, \ldots, f_n$.
Fix some $\epsilon > 0$. Because $f_n \to f$ uniformly on S, fix some N where any $x \in S$ gives $|f(x) - f_{N+1}(x)| < \epsilon$. Then M_{N+1} is a bound for all $f_1, f_2, \ldots, f_{N+1}$. Further, notice:

$$|f(x) - f_{N+1}(x)| < \epsilon \iff f_{N+1}(x) - \epsilon < f(x) < f_{N+1}(x) + \epsilon \implies |f(x)| < f_{N+1}(x) + \epsilon \leq M_n + \epsilon,$$

so we conclude that f is a bounded function on S as we have shown that for all $x \in S$, $|f(x)| < M_n + \epsilon$. □

Definition: Uniformly Cauchy -

Sequence (f_n) is **uniformly Cauchy** if $\forall_{\epsilon>0}\exists_N : \forall_{x\in S}\forall_{m,n>N}, \qquad |f_n(x) - f_m(x)| < \epsilon$.

Theorem 0.3. Uniformly Cauchy (f_n) implies $f_n \to f$ uniformly on S (there exists some such f).

Theorem 0.4. If $\sum_{k=0}^{\infty} g_k$ is uniformly Cauchy on S, then the series converges uniformly on S.

Theorem 0.5. Consider a series $\sum_{k=0}^{\infty}$ of functions on $S \subset \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S. Informally, I say 'the (infinite) sum of commonly uniformly convergent functions is continous'.

Theorem 0.6. Weierstrass M-test. Let (M_k) be a sequence of **nonnegative** real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all $x \in S$, then $\sum g_k$ converges uniformly on S.

Problem 25.6.

1. Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.
2. Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1,1]$?

Solution. (1) Suppose $\sum |a_k| = M < \infty$, so that $|a_k| < M_k$ for all $k \in \mathbb{N}$ and $(M_k) \to M$. Then $(|a_k|)$ is a sequence of nonnegative real numbers where $\sum |a_k| < \infty$ (given). By the Weierstrass M-test, we conclude $\sum a_k x^k$ converges uniformly to some function f on $[-1,1]$. It remains to show f is continuous.

Notice that for all $x \in [-1,1]$, we have $\sum a_k x^k \leq |a_k| < M_k$ for all k, and the partial sums of $\sum a_k x^k$ are polynomials so are continuous on $[-1,1]$. Hence by the theorem that states that the infinite sum of commonly uniformly converging continuous functions is continuous (theorem 0.5 on the previous page), we conclude $f := \sum a_k x^k$ is continuous, and we are done.

(2) Because $\sum \left|\frac{1}{n^2}\right|$ is a known convergent sequence (if this is imprecise, we cite $p = 2$ test), invoking part (1) above directly shows that indeed yes, $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represents a continuous function on $[-1,1]$.

□

Problem 25.7. Show $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

Solution. Assuming we do not know the series expansion of $\cos(nx)$ (and thus do not invoke the above result), we (ab)use another property of $\cos x$, we know that $|\cos(nx)| \leq 1$ for all $n, x \in \mathbb{R}$, so take

$$f_n(x) := \frac{1}{n^2} \cos(nx),$$

so that

$$|f_n(x)| \leq |n^{-2}| \leq 1, \quad \forall_{x \in \mathbb{R}, n \in \mathbb{N}}.$$

Because $\sum \frac{1}{n^2}$ converges (via $p = 2$ test), we conclude that the Weierstrass M-test gives our desired result: $\sum f_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$. Notice we do not claim the function to which this converges, although this can be reasoned by pointwise convergence and periodicity. □

Theorem 0.7. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with $R > 0$ (possibly $+\infty$). If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Theorem 0.8. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad |x| < R$$

Theorem 0.9. Abel's Theorem. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then f is continous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.

Problem 26.2.

1. Observe $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1 [page 211 in the book].
2. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Compare with Exercise 14.13(d).
3. Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

For part (2), we include Exercise 14.13(d), which builds on Exercise 14.13(c), which uses a fact generalized from Exercise 14.13(b). '(c) Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$; hint: $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$. (d) Use (c) to calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.'

Solution. (1) This identity is a neat consequence of infinite series, where the differentiation operator would otherwise drop a degree for a finite-degree polynomial. Consider the canonical geometric series, where we have:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

Taking the derivative of both sides yields:

$$-(1-x)^{-2}(-1) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

and multiplying across by x gives:

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots = \sum_{n=1}^{\infty} nx^n,$$

where the series index starts from 1 (which is as we wished to show).

(2) Now we evaluate $\sum_{n=1}^{\infty}$ by setting $x := 1/2$ in the expression from (1) above. That is,

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n|_{x:=1/2} = \frac{1/2}{(1-1/2)^2} = \boxed{2}.$$

(3) Similarly, we simply set $x := \frac{1}{3}$ and $x := \frac{-1}{3}$ into our expression in (1) to get our desired results:

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} nx^n|_{x:=1/3} = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{4/9} = \boxed{\frac{3}{4}},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \sum_{n=1}^{\infty} nx^n|_{x:=-1/3} = \frac{(-1/3)}{(1+1/3)^2} = \frac{-1/3}{16/9} = \boxed{\frac{-3}{16}}.$$

□

Problem 26.3.

1. Use Exercise 26.2 to derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$.
2. Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

Solution. (1) From the above, we derived (and actually were simply informed) that:

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}, \qquad \forall_{x\in\mathbb{R}:|x|<1}$$

Taking the derivative (with respect to x) across the equations yields:

$$x'\left(\frac{1}{(1-x)^2}\right) + x\left[(1-x)^{-2}\right]' = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Multiplying across by x and simplifying the LHS gives:

$$x\left[\frac{1}{(1-x)^2}+\frac{(-2)(-1)x}{(1-x)^3}\right]=\sum_{n=1}^{\infty} n^2x^n$$

$$=\frac{x(1-x)+2x}{(1-x)^3}=\frac{x-x^2+2x^2}{(1-x)^3}=\boxed{\frac{x^2+x}{(1-x)^3}}.$$

(2) Now we use this expression to evaluate $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$. Setting $x:=1/2$ gives:

$$\sum_{n=1}^{\infty} n^2x^n|_{x:=1/2}=\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\frac{(1/2)^2+(1/2)}{(1/2)^3}=\frac{3/4}{1/8}=\boxed{6},$$

and setting $x:=1/3$ gives:

$$\sum_{n=1}^{\infty} n^2x^n|_{x:=1/3}=\sum_{n=1}^{\infty}\frac{n^2}{3^n}=\frac{(1/3)^2+(1/3)}{(2/3)^3}=\frac{4/9}{8/27}=\boxed{\frac{3}{2}}.$$

□

Problem 26.7. Let $f(x)=|x|$ for $x\in\mathbb{R}$. Is there a power series $\sum a_nx^n$ such that $f(x)=\sum_{n=0}^{\infty}a_nx^n$ for all x? Discuss. [Max's note: you should even think about whether there is a power series representing f on any interval around 0 whatsoever].

Solution. This function is a canonical example of a 'simple' continuous function but not differentiable (at $x=0$). Because $f(x):=|x|$ is not differentiable at $x=0$, $f(x)=|x|$ cannot be represented by a power series $\sum_{n=0}^{\infty}a_nx^n$. Because any radius of convergence for a power series (as our first theorem in Ross dictates) must include its center (in this case 0) and some radius $R>0$, we conclude that $\boxed{\text{there is no power series with this desired property}}$. To be precise, there is no poower series $\sum a_nx^n$ with the property that $f(x)=\sum_{n=0}^{\infty}a_nx^n$ for all x.
Another nice example is given in Ross on Page 204:

204 **4. Sequences and Series of Functions**

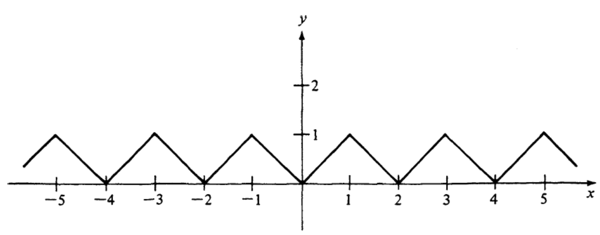

FIGURE 25.1

Example 3
Let g be the function drawn in Fig. 25.1, and let $g_n(x)=g(4^nx)$ for all $x\in\mathbb{R}$. Then $\sum_{n=0}^{\infty}(\frac{3}{4})^ng_n(x)$ is a series of functions. The limit function f is continuous on $\mathbb{R}$, but has the amazing property that it is not differentiable at any point! The proof of the nondifferentiability of f is somewhat delicate; see [62, 7.18]. A similar example is given in Example 38.1 on page 348. □

□