Math 104: Homework 8/9

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

23.1 For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(b) $\sum(\frac{x}{n})^n$
Solution: Here we have $a_n = n^{-n}$, so we see $|a_n|^{1/n} = n^{-1} \to 0$, meaning the radius of convergence is $R = \infty$, so the interval of convergence is $\mathbb{R}$.

(e) $\sum(\frac{2^n}{n!})x^n$
Solution: Here we have

$$|a_n|^{1/n} = \frac{2}{(n!)^{1/n}},$$

and we have seen that $(n!)^{1/n} \to \infty$ (this is because $(n+1)!/n! = n+1 \to \infty$ and $\liminf |a_n|^{1/n} \geq \liminf |a_{n+1}/a_n|$), so $|a_n|^{1/n} \to 0$, meaning again $R = \infty$ and the series converges everywhere.

(h) $\sum(\frac{(-1)^n}{n^2 \cdot 4^n})x^n$
Solution: Here we have

$$|a_n|^{1/n} = \frac{1}{4n^{2/n}} \to \frac{1}{4},$$

meaning $\beta = 1/4$, so $R = 4$. When we check $x = -4$ we see that the series becomes

$$\sum \frac{1}{n^2},$$

which converges (since series $\sum 1/n^p$ converge when $p > 1$) and when $x = 4$ the series is

$$\sum \frac{(-1)^n}{n^2},$$

which also converges by the Alternating Series Test. So the interval of convergence is $[-4, 4]$.

23.2 Repeat Exercise 23.1 for the following:

(c) $\sum x^{n!}$
Solution: Here we see that $a_n = 1$ if $n = k!$ for some k and $a_n = 0$ otherwise. This means $\limsup |a_n|^{1/n} = 1$ because the value 1 is repeated infinitely many times. So we have $R = 1$. When $x = \pm 1$ the series clearly diverges because in both cases the terms fail to converge to zero. So the interval of convergence is $(-1, 1)$.

(d) $\sum \frac{3^n}{\sqrt{n}}x^{2n+1}$
Solution: In this case we see that we have $a_n = 3^{(n-1)/2}/\sqrt{(n-1)/2}$ when $n = 2k+1$ for some $k \in \mathbb{N}$, while $a_n = 0$ otherwise. This shows that

$$\limsup |a_n|^{1/n} = \lim_{n\to\infty} \frac{3^{(n-1)/(2n)}}{[(n-1)/2]^{1/(2n)}} = \sqrt{3},$$

so $R = 1/\sqrt{3}$. When $x = -1/\sqrt{3}$, the series becomes

$$\sum \frac{(-1)^n}{\sqrt{3}\sqrt{n}},$$

which converges by the Alternating Series Test. When $x = 1/\sqrt{3}$ the series is

$$\sum \frac{1}{\sqrt{3}\sqrt{n}},$$

which diverges because $\sum 1/\sqrt{n}$ diverges.

23.4 For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.

(a) Find $\limsup (a_n)^{1/n}$, $\liminf (a_n)^{1/n}$, $\limsup |\frac{a_{n+1}}{a_n}|$ and $\liminf |\frac{a_{n+1}}{a_n}|$.
Solution: We have

$$a_n^{1/n} = \frac{4+2(-1)^n}{5} = \begin{cases} 2/5 & n \text{ odd} \\ 6/5 & n \text{ even} \end{cases}$$

so we see $\limsup (a_n)^{1/n} = 6/5$ and $\liminf (a_n)^{1/n} = 2/5$.
On the other hand, we have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left(\frac{4+2(-1)^{n+1}}{5}\right)\left(\frac{((4+2(-1)^{n+1})/5)^n}{((4+2(-1)^n)/5)^n}\right) \\ &= \left(\frac{4+2(-1)^{n+1}}{5}\right)\left(\frac{4+2(-1)^{n+1}}{4+2(-1)^n}\right)^n, \end{aligned}$$

meaning

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \frac{6}{5}\cdot 3^n & n \text{ odd} \\ \frac{4}{5}\left(\frac{1}{3}\right)^n & n \text{ even} \end{cases}$$

so we see $\limsup |\frac{a_{n+1}}{a_n}| = \infty$ and $\liminf |\frac{a_{n+1}}{a_n}| = 0$.

(b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.
Solution: The series $\sum a_n$ diverges by part (a), using the Root Test, since $\limsup |a_n|^{1/n} > 1$. Similarly, we see that $\sum (-1)^n a_n$ diverges by the Root Test because $|(-1)^n a_n|^{1/n} = a_n^{1/n}$.

(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence of the series.
Solution: Since $\limsup |a_n|^{1/n} = 6/5$ we see that $R = 5/6$. When $x = -5/6$ we see that the terms $a_n x^n = 1$ when n is even, so the series cannot converge. Similarly when $x = 5/6$ we have that $a_n x^n = 1$ when n is even, so the series doesn't converge here either. So the interval of convergence of the series is $(-5/6, 5/6)$.

24.2 For $x \in [0, \infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.
Solution: For any fixed value x, we see that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x/n = 0$, so $f_n \to 0$ pointwise, i.e. $f(x) = 0$.

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

Solution: We claim that, indeed, $f_n \to 0$ uniformly on $[0, 1]$. To see this, let $\varepsilon > 0$. Then take $N > 1/\varepsilon$, so that for $n > N$ we have $1/n < \varepsilon$. Then we see that, for all $n > N$ and for all $x \in [0, 1]$, we have

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n} \leq \frac{1}{n} < \varepsilon,$$

showing that the convergence is uniform.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution: No, the convergence of (f_n) is *not* uniform on $[0, \infty)$. To see this, take $\varepsilon = 1$. We claim that, in fact, for every $n \in \mathbb{N}$ we can find x such that $|f_n(x) - 0| > \varepsilon$, which is certainly enough to show that the convergence is not uniform. This is because we can simply take $x = 2n$, in which case $|f_n(x)| = 2 > 1 = \varepsilon$, proving the claim.

24.3 Repeat exercise 24.2 for $f_n = \frac{1}{1+x^n}$.

Solution: For part (a), we see that when $x < 1$ we have $f_n(x) \to 1$, and when $x = 1$ we have $f_n(x) = 1/2$ for each n so $f_n(x) \to 1/2$, and lastly for $x > 1$ we have $f_n(x) \to 0$. So the function f is given by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1. \end{cases}$$

For part (b), note that the restriction of f to the interval $[0, 1]$ is discontinuous (specifically f is discontinuous at $x = 1$), so we can conclude by 24.3 that (f_n) cannot converge uniformly to f on this interval, because otherwise the limit would be continuous (since the f_n are all clearly continuous). To make this more explicit, take $\varepsilon = 1/4$. Note that when we have $(1/3)^{1/n} < x < 1$ we have that $1/3 < x^n$ so that $4/3 < 1 + x^n$, meaning $3/4 > 1/(1 + x^n) = f_n(x)$. So for each n, we have that whenever $x \in ((1/3)^{1/n}, 1)$ we have

$$|f_n(x) - f(x)| = |f_n(x) - 1| = 1 - f_n(x) > 1/4 = \varepsilon,$$

since $f_n(x) < 3/4$ in this interval. This means the convergence is not uniform.

For part (c), we trivially have that (f_n) does not converge uniformly on $[0, \infty)$ because this would imply uniform convergence on any subset of $[0, \infty)$, including $[0, 1]$, and we just showed the convergence is not uniform on $[0, 1]$.

25.5 Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

Solution: Suppose for the sake of contradiction that f is not bounded. Then we can take a sequence of points $x_n \in S$ where $f(x_n) > n$ for each $n \in \mathbb{N}$. Now take $\varepsilon = 1$. Since $f_n \to f$ uniformly, we can find N for which $|f_n(x) - f(x)| < \varepsilon$ for every $n > N$ and every $x \in S$. However, we claim this means f_{N+1} is in fact unbounded. To see this, note that by our choice of N we must in particular have that $|f_{N+1}(x_k) - f(x_k)| < \varepsilon = 1$ for every $k \in \mathbb{N}$. But since $f(x_k) > k$ this means that $f_{N+1}(x_k) > k - 1$ for every k. But this demonstrates that the values $f_{N+1}(x_k)$ grow without bound as k increases, meaning f_{N+1} is unbounded. So if all the f_n are assumed bounded we must indeed have that f is bounded as well.

25.6 (a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.

Solution: Note that in this case we can use the Weierstrass M-test and set $M_k = |a_k|$, so that $|a_k x^k| \leq M_k$ for $x \in [-1,1]$, meaning that indeed $\sum a_k x^k$ converges uniformly on $[-1,1]$. The fact that the partial sums of $\sum a_k x^k$ are continuous functions on $[-1,1]$, and that the convergence is uniform, shows by 24.3 that the limit $\sum a_k x^k$ is continuous on $[-1,1]$.

(b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1,1]$?

Solution: Yes, $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ is continuous on $[-1,1]$ by part (a) since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

25.7 Show $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

Solution: Take $g_k(x) = \frac{1}{k^2} \cos kx$ (defined on $\mathbb{R}$) and set $M_k = \frac{1}{k^2}$. Then we see that $\sum M_k < \infty$ and $|g_k(x)| \leq M_k$ for each k, so by the Weierstrass M-test the series converges uniformly on $\mathbb{R}$. Because the g_k (and hence the partial sums) are continuous and the convergence is uniform, the limit is continuous as well by 24.3.

26.2 (a) Observe $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1 [page 211 in the book].

Solution: In example 1, we see

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

so it follows that $\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$, as claimed.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Compare with Exercise 14.13(d).

Solution: Using the formula above, we see that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is an example of the above power series where $x = 1/2$, so we have

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2,$$

which agrees with the value we obtained in 14.13(d) (at least after I corrected the typo/arithmetic error in the solutions).

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

Solution: We have

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-1/3}{(1-(-1/3))^2} = -\frac{3}{16}.$$

26.3 (a) Use Exercise 26.2 to derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$.

Solution: Since we have $\sum nx^n = x/(1-x)^2$, we see that by taking the derivative once more on both sides of the equation, we have

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3},$$

so multiplying by x on both sides shows

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x + x^2}{(1-x)^3}.$$

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

Solution: We have

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1/2 + (1/2)^2}{(1-1/2)^3} = 6,$$

and

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{1/3 + (1/3)^2}{(1-1/3)^3} = \frac{3}{2}.$$

26.7 Let $f(x) = |x|$ for $x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x? Discuss. [Max's note: you should even think about whether there is a power series representing f on any interval around 0 whatsoever.]

Solution: No, there is no power series $\sum a_n x^n$ representing $f(x) = |x|$ on $\mathbb{R}$. This is because any such power series would necessarily represent a differentiable function on $\mathbb{R}$ by 26.5, and f is not differentiable at $x = 0$. In general, there can be no power series representation $\sum a_n x^n$ for f on any interval $(-R, R)$, or even more generally no power series $\sum a_n (x-c)^n$ on any interval $(c-R, c+R)$ containing 0. We do, however, have that $c + (x-c)$ is (when its domain is appropriately restricted) a power series representation of f on the interval $(0, 2c)$ when $c > 0$, and $-c - (x-c)$ is (again, when its domain is appropriately restricted) a power series representation of f on the interval $(2c, 0)$ when $c < 0$.