Math 104: Homework 7

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

20.1 Sketch the function $f(x) = \frac{x}{|x|}$. Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to 0}$ *when they exist*. Also indicate when they do not exist.

Solution: Here we see that $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to 0^+} f(x) = 1$, and $\lim_{x\to-\infty} f(x) = -1$. I also appear to have neglected $\lim_{x\to 0^-} f(x)$ in the problem statement above, but that should be $\lim_{x\to 0^-} f(x) = -1$, which in turn means $\lim_{x\to 0} f(x)$ does not exist.

20.3 Repeat exercise 20.1 for $f(x) = \frac{\sin x}{x}$. See Example 9 of §19 [page 149 in the textbook].

Solution: From inspection we see $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0} f(x) = 1$.

20.5 Prove the limit assertions in Exercise 20.1.

Solution: To show $\lim_{x\to\infty} f(x) = 1$, we must show that for all $\varepsilon > 0$ we can find M such that for $x > M$ we have $|f(x) - 1| < \varepsilon$. This is straightforward as for any such ε we can simply take $M = 1$ (or indeed any $M > 0$) and see that for $x > M$ we have $|f(x) - 1| = 0 < \varepsilon$.

To show $\lim_{x\to 0^+} f(x) = 1$ we must show that for all $\varepsilon > 0$ we can find $\delta > 0$ so that for $0 < x < \delta$ we have $|f(x) - 1| < \varepsilon$. This is also straightforward as we can simply take $\delta = 1$ for any such ε and see that for $0 < x < \delta$ we have $|f(x) - 1| = 0 < \varepsilon$.

For $\lim_{x\to-\infty} f(x) = -1$ we must show that for all $\varepsilon > 0$ we can find M so that for $x < M$ we have $|f(x) + 1| < \varepsilon$. In this case we can simply take $M = -1$ for any such ε, and for $x < M$ we will have $|f(x) + 1| = 0 < \varepsilon$.

For $\lim_{x\to 0^-} f(x) = -1$ we show that for all $\varepsilon > 0$ we can find δ such that for $-\delta < x < 0$ we have $|f(x) + 1| < \varepsilon$. Here we can take $\delta = -1$ for any such ε and have $|f(x) + 1| = 0 < \varepsilon$ for $-\delta < x < 0$.

To show $\lim_{x\to 0} f(x)$ does not exist, we could appeal to Theorem 20.10, or we could argue directly as follows: suppose there were some number L such that $\lim_{x\to 0} f(x) = L$. Then by definition we would have to have for any sequence (x_n) with $x_n \in (a, b) \setminus \{0\}$ (for some $a < 0 < b$) that $\lim f(x_n) = L$. However, we see this is impossible because for any such a and b we can take one sequence $x_n = \min\{1/n, b/2\}$ and a second sequence $x'_n = \max\{-1/n, a/2\}$ and see that clearly $x_n \in (a, b) \setminus \{0\}$ and $x'_n \in (a, b) \setminus \{0\}$ for all n, and we also have $x_n \to 0$ and $x'_n \to 0$, but $f(x_n) = 1$ for all n so $f(x_n) \to 1$ while $f(x'_n) = -1$ for all n so $f(x'_n) \to -1$, contradicting $\lim f(x_n) = \lim f(x'_n) = L$.

20.7 Prove the limit assertions in Exercise 20.3.

Solution: To see $\lim_{x\to\infty} f(x) = 0$, let $\varepsilon > 0$. Then choose $M > 1/\varepsilon$. Then for $x > M$ we have

$$|f(x) - 0| = \left|\frac{1}{x}\right| |\sin x| \le \left|\frac{1}{x}\right| < \frac{1}{M} < \varepsilon,$$

showing $\lim_{x\to\infty} f(x) = 0$ as desired.

For $\lim_{x\to 0^+} f(x) = 1$, let $\varepsilon > 0$. By the assumption that $\tilde{h}(x)$ in Example 9 is continuous, we can invoke the ε-δ property of continuity to select δ so that for $|x| < \delta$ we have $|\tilde{h}(x) - 1| < \varepsilon$.

Then in particular this inequality holds when $0 < x < \delta$ and since $f(x) = \tilde{h}(x)$ for these values we see we that $|f(x) - 1| < \varepsilon$ as well. In fact, this same argument can be used to show that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 1$.

For $\lim_{x\to 0^-} f(x) = 0$, let $\varepsilon > 0$ and choose $M < -1/\varepsilon$. Then for $x < M$ we have

$$|f(x) - 0| = \left|\frac{1}{x}\right| |\sin x| \leq \left|\frac{1}{x}\right| = -\frac{1}{x} < -\frac{1}{M} < \varepsilon,$$

where the latter inequalities hold because $x < M$ implies $1/x > 1/M$ implies $-1/x < -1/M$, as well as the fact that $M < -1/\varepsilon$ implies $-M > 1/\varepsilon$ implies $-1/M < \varepsilon$.

20.12 (a) Sketch the function $f(x) = (x-1)^{-1}(x-2)^{-2}$.

Solution:

(b) Determine $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

Solution: We claim $\lim_{x\to 2^+} f(x) = \infty$. To see this, let $M > 0$. Then for $\delta > 0$, and also taking $\delta < 1$, we have for $2 < x < 2 + \delta$

$$(x-1)^{-1}(x-2)^{-2} > (1+\delta)^{-1}\delta^{-2} > \frac{\delta^{-2}}{2}$$

so we see that by choosing $\delta < \sqrt{1/(2M)}$ as well we can force the lattermost expression above to be greater than M, which means taking $\delta < \min\{1, \sqrt{1/(2M)}\}$ we can force $f(x) > M$ for $2 < x < 2 + \delta$. So by Discussion 20.9 we have shown $\lim_{x\to 2^+} f(x) = \infty$.

We claim $\lim_{x\to 2^-} = \infty$. To see this, let $M > 0$. We see that for any choice of δ with $0 < \delta < 1$ we have, for $2 - \delta < x < 2$, that

$$(x-1)^{-1}(x-2)^{-2} > (2-1)^{-1}(2-\delta-2)^{-2} = \delta^{-2},$$

so by taking $\delta < \min\{1, \sqrt{1/M}\}$ we can force $f(x) > M$. So again by Discussion 20.9 we have shown $\lim_{x\to 2^-} f(x) = \infty$.

We claim $\lim_{x\to 1^+} f(x) = \infty$. To see this, let $M > 0$. We see that for any choice of δ with $0 < \delta < 1$ and for $1 < x < 1 + \delta$ we have $(x-2)^{-2} > 1$ so that

$$(x-1)^{-1}(x-2)^{-2} > \delta^{-1},$$

so by taking $\delta < \min\{1, 1/M\}$ we see we can force $f(x) > M$. So again by Discussion 20.9 we have shown $\lim_{x\to 1^+} f(x) = \infty$.

We claim $\lim_{x\to 1^-} f(x) = -\infty$. To see this, Let $M < 0$. Then for $0 < \delta < 1$ and $1 - \delta < x < 1$ we have $(x-2)^{-2} > \frac{1}{4}$, so

$$(x-1)^{-1}(x-2)^{-2} < -\frac{\delta^{-1}}{4},$$

so by taking $\delta < \min\{1, 1/(4M)\}$ we see we can force $f(x) < M$, so by Discussion 20.9 we have shown $\lim_{x\to 1^-} f(x) = -\infty$.

(c) Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to 1} f(x)$ if they exist.

Solution: By Theorem 20.10, since $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ both exist and share a common value of ∞, we have $\lim_{x\to 2} f(x) = \infty$ as well. On the other hand, by the same theorem, since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, we have that $\lim_{x\to 1} f(x)$ does not exist.

20.19 The limits defined in Definition 20.3 do not depend on the choice of the set S. As an example, consider $a < b_1 < b_2$ and suppose f is defined on (a, b_2). Show that if the limit $\lim_{x \to a^S} f(x)$ exists for either $S = (a, b_1)$ or $S = (a, b_2)$, then the limit exists for the other choice of S and these limits are identical. Their common value is what we write as $\lim_{x \to a^+} f(x)$.

Solution: Suppose $\lim_{x \to a^S} f(x)$ exists for $S = (a, b_2)$, and say its value is L. Now let (x_n) be a sequence of points in $S' = (a, b_1)$ with $x_n \to a$. Then since $b_1 < b_2$ we have $(a, b_1) \subseteq (a, b_2)$, so (x_n) is a sequence of points in (a, b_2) as well (with $x_n \to a$). So since $\lim_{x \to a^S} f(x) = L$ we have $\lim f(x_n) = L$. But since (x_n) was arbitrary in (a, b_1) this shows that $\lim_{x \to a^{S'}} f(x) = L$ as well.

On the other hand, suppose $\lim_{x \to a^S} f(x) = L$ where $S = (a, b_1)$ and let (x_n) be a sequence of points in $S' = (a, b_2)$ with $x_n \to a$. Since $x_n \to a$ we can find N such that for $n \geq N$ we have $a < x_n < b_1$. Then the sequence $(x_n)_{n=N}^{\infty}$ is a sequence in S converging to a so since $\lim_{x \to a^S} f(x) = L$ we have $\lim f(x_n) = L$, where the limit is technically being taken over $n \geq N$. However, this also implies $\lim f(x_n) = L$ since the limit of the sequence $(f(x_n))$ does not depend on its first N terms. So since (x_n) was an arbitrary sequence in S' converging to a we have shown $\lim_{x \to a^{S'}} f(x) = L$ in this case as well. So in any case if one of the two limits exist, the other does as well and they have the same value.

21.10 Show there exist continuous functions

(a) Mapping $(0, 1)$ onto $[0, 1]$,

Solution: Consider the function $f(x) = \frac{1}{2} + \frac{1}{2}\sin(2\pi x)$. Then clearly $f(x) \in [0, 1]$ for $x \in (0, 1)$ because $-1 \leq \sin(2\pi x) \leq 1$. Furthermore, we have $f(\frac{1}{4}) = 1$ and $f(\frac{3}{4}) = 0$, so clearly f achieves a maximum of 1 at $x = 1/4$ and a minimum of 0 at $x = 3/4$, and by the intermediate value theorem f must achieve every value in between 0 and 1, so $f((0, 1)) = [0, 1]$ as desired.

(b) Mapping $(0, 1)$ onto $\mathbb{R}$,

Solution: Take $f(x) = \tan(\pi(x - \frac{1}{2}))$. Then we have $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = -\infty$ so by the Intermediate Value Theorem we have $f((0, 1)) = \mathbb{R}$ as desired.

(c) Mapping $[0, 1] \cup [2, 3]$ onto $[0, 1]$.

Solution: Let

$$f(x) = \begin{cases} x & x \in [0, 1] \\ x - 2 & x \in [2, 3] \end{cases}$$

Then it is straightforward that $f([0, 1]) = [0, 1]$ and $f([2, 3]) = [0, 1]$ so $f([0, 1] \cup [2, 3]) = [0, 1]$ as desired.

(d) Explain why there are no continuous functions mapping $[0, 1]$ onto $(0, 1)$ or $\mathbb{R}$.

Solution: Continuous functions mapping $[0, 1]$ to $(0, 1)$ or $\mathbb{R}$ because the former is a compact set, while the latter are not (by the Heine-Borel theorem), and the image of a compact set under a continuous function must always be compact.

22.1 Show there do not exist continuous functions

(a) Mapping $[0, 1]$ onto $[0, 1] \cup [2, 3]$,

Solution: There is no continuous function mapping $[0, 1]$ onto $[0, 1] \cup [2, 3]$ because the former is a connected set while the latter is not. To see that the latter set $E = [0, 1] \cup [2, 3]$

is disconnected it suffices to take $U_1 = (-1, 3/2)$ and $U_2 = (3/2, 4)$. Then clearly $U_1 \cap U_2 = \emptyset$ so $(U_1 \cap E) \cap (U_2 \cap E) = \emptyset$, and we have $U_1 \cap E = [0, 1]$ and $U_2 \cap E = [2, 3]$ so indeed $E = (U_1 \cap E) \cup (U_2 \cap E)$, and also $U_1 \cap E \neq \emptyset$ and $U_2 \cap E \neq \emptyset$.

(b) Mapping $(0, 1)$ onto $\mathbb{Q}$.

Solution: We have again that $(0, 1)$ is connected but $\mathbb{Q}$ is not, so a continuous function mapping $(0, 1)$ onto $\mathbb{Q}$ cannot exist. To see that $\mathbb{Q}$ is disconnected, it suffices to take $U_1 = (-\infty, \sqrt{2})$ and $U_2 = (\sqrt{2}, \infty)$. Then clearly $U_1 \cap U_2 = \emptyset$ so indeed $(U_1 \cap \mathbb{Q}) \cap (U_2 \cap \mathbb{Q}) = \emptyset$. Also since $\sqrt{2}$ is not rational we have for any $r \in \mathbb{Q}$ that either $r < \sqrt{2}$ or $r > \sqrt{2}$ so either $r \in \mathbb{Q} \cap U_1$ or $r \in \mathbb{Q} \cap U_2$, meaning $\mathbb{Q} = (\mathbb{Q} \cap U_1) \cup (\mathbb{Q} \cap U_2)$. Lastly we clearly have that $\mathbb{Q} \cap U_1$ and $\mathbb{Q} \cap U_2$ are nonempty.

22.5 Let E and F be connected sets in some metric space.

(a) Prove that if $E \cap F \neq \emptyset$, then $E \cup F$ is connected.

Solution: Here is the basic idea: if E and F are connected and $E \cap F = \emptyset$, then (in many cases – though not always, since it is possible for $E \cap F = \emptyset$ but $E \cup F$ is still connected) it is easy to disconnect $E \cup F$ by simply taking U_1 to be an open set containing only E and U_2 to be an open set containing only F, thus separating them. However, when we force $E \cap F \neq \emptyset$, what happens is that if U_1 and U_2 have to cover $E \cup F$ then the part where E and F intersect gets "dragged" into either U_1 or U_2, so that we either end up having that U_1 and U_2 each contain a piece of E or each contain a piece of F, which results in them disconnecting either E or F accordingly (which would contradict the assumption that E and F are connected). Here is how we make this argument formal:

Suppose $E \cup F$ is disconnected and $E \cap F \neq \emptyset$. Then we will show either E or F is disconnected. Let U_1 and U_2 be open sets disconnected $E \cup F$. Then there are two cases: either E is contained entirely inside one of the two open sets U_1 and U_2 or not. Let's treat the latter case first: here we claim that U_1 and U_2 disconnect E. We first note that

$$(E \cap U_1) \cap (E \cap U_2) \subseteq ((E \cup F) \cap U_1) \cap ((E \cup F) \cap U_2) = \emptyset,$$

so indeed $(E \cap U_1) \cap (E \cap U_2) = \emptyset$. We also note that the assumption we are making in this case is literally that $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$, so the only other thing to verify is that $E = (E \cap U_1) \cup (E \cap U_2)$. Clearly the right-hand side is always a subset of the left, so we need only show containment in the other direction. Let $x \in E$. Then $x \in E \cup F$, so either $x \in (E \cup F) \cap U_1$ or $x \in (E \cup F) \cap U_2$. But this means either $x \in U_1$ or $x \in U_2$, which combined with $x \in E$ in turn means $x \in E \cap U_1$ or $x \in E \cap U_2$, so indeed $E = (E \cap U_1) \cup (E \cap U_2)$.

Now we consider the other case: say $E \subseteq U_1$ without loss of generality. Then we claim that U_1 and U_2 disconnect F. To see this, note that since $E \cap F \neq \emptyset$, we can find $x \in E \cap F$. But then since $x \in E$ we have $x \in U_1$, so $F \cap U_1 \neq \emptyset$. On the other hand, we also can't have $F \subseteq U_1$ because then we would have $E \cup F \subset U_1$, which would mean that $(E \cup F) \cap U_1 = E \cup F$, and since we have $((E \cup F) \cap U_1) \cap ((E \cup F) \cap U_2) = \emptyset$ this would imply $(E \cup F) \cap U_2 = \emptyset$, contradicting that U_1 and U_2 disconnect $E \cup F$. So we must also have $F \cap U_2 \neq \emptyset$. Then an argument similar to the previous case shows that U_1 and U_2 disconnect F.

(b) Give an example to show $E \cap F$ need not be connected. Incidentally, the empty set **is** connected.

Solution: Take $E = \{(\cos\theta, \sin\theta) \,:\, \theta \in [-2\pi/3, 2\pi/3]\}$ and $F = \{(\cos\theta, \sin\theta) \,:\, \theta \in [\pi/3, 5\pi/3]\}$. We can argue that E and F are each connected because they are the image of connected sets in $\mathbb{R}$ under the continuous function $\gamma : \mathbb{R} \to \mathbb{R}^2$ sending $\theta \mapsto (\cos\theta, \sin\theta)$ (this function is continuous under the assumption that cos and sin are each individually continuous). On the other hand, we have that

$$E \cap F = \{(\cos\theta, \sin\theta) \,:\, \theta \in [\pi/3, 2\pi/3]\} \cup \{(\cos\theta, \sin\theta) \,:\, \theta \in [4\pi/3, 5\pi/3]\},$$

which can be disconnected by the open subsets $U_1 = \{(x, y) \,:\, y > 0\}$ and $U_2 = \{(x, y) \,:\, y < 0\}$ in $\mathbb{R}^2$.