Math 104, Summer 2019

PSET #7 (due Thurs 7/25/2019)

General Definition 20.1: Limit of a function -

Let $S \subset \mathbb{R}$, and $a \in \mathbb{R} \cup \{\infty, -\infty\}$, where a is a subsequential limit in S (if $a \notin S$, we need a cluster of points near a so that we can get a sequence to get arbitrarily close to a). Let $L \in \mathbb{R} \cup \{\pm\infty\}$.
Then we write

$$\lim_{x \to a^S} f(x) = L,$$

if f is a function defined on S and for **every** sequence (x_n) in S with $x_n \to a$, we have $\lim_{n\to\infty} f(x_n) = L$.

Problem 20.1. Sketch the function $f(x) = \frac{x}{|x|}$. Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to 0}$ **when they exist**. Also indicate when they do not exist.

Problem 20.5. Prove the limit assertions in Exercise 20.1.

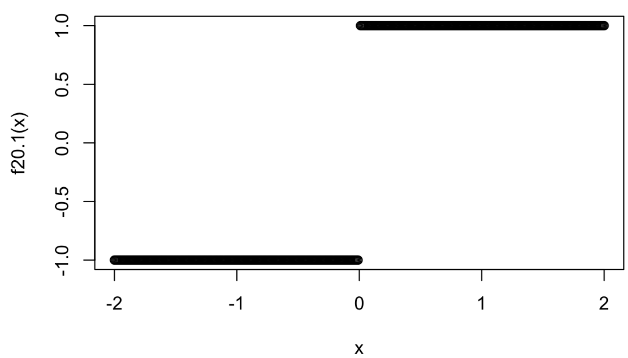

Figure 1: The graph of function x/abs(x). Notice the 'jump' discontinuity.

Solution. By inspection, we claim:

$$\lim_{x\to-\infty} f(x) = -1 \qquad \lim_{x\to 0^-} f(x) = -1 \qquad \lim_{x\to 0^+} f(x) = 1 \qquad \lim_{x\to\infty} f(x) = 1$$

To see this precisely, let $S := (0, \infty)$, so that $\forall_{x\in S} f(x) = 1$. Because the value of $f(x)$ is exactly 1 for all such $x \in S$, we conclude:

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(x) = 1.$$

Similarly, set $\hat{S} := (-\infty, 0)$, so that $\forall_{x\in\hat{S}} f(x) = -1$. Because $f(x)$ is exactly -1 for all $x \in \hat{S}$, we conclude:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to-\infty} f(x) = -1.$$

Because $\lim_{x\to 0^-} f(x) = -1 \neq 1 = \lim_{x\to 0^+} f(x)$, we conclude $\lim_{x\to 0} f(x)$ does not exist.

$\square$

Problem 20.3. Repeat exercise 20.1 for $f(x) = \frac{\sin x}{x}$. See Example 9 of §19 [page 149 in the textbook].

Problem 20.7. Prove the limit assertions in Exercise 20.3.

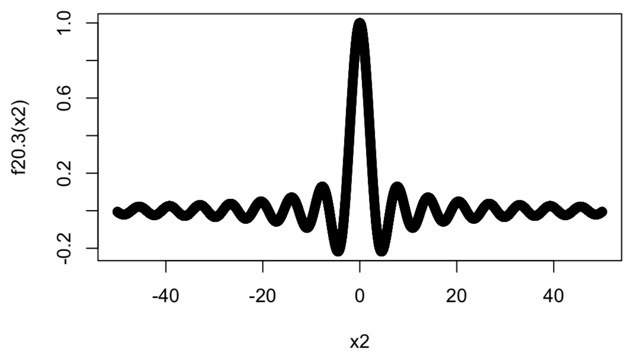

Figure 2: Plot of (1/x)sin(x)

Solution. It appears:

$$\lim_{x\to-\infty} f(x) = 0 \qquad \lim_{x\to 0^-} f(x) = 1 \qquad \lim_{x\to 0^+} f(x) = 1 \qquad \lim_{x\to\infty} f(x) = 0$$

To see the limits at $\pm\infty$ precisely, take and fix any sequence (s_n) in $(0,\infty)$ with $s_n \to \infty$. Then $a_n := \frac{1}{s_n}$ has the property $a_n \to 0$. Because $|\sin(x)| \leq 1$ and is thus bounded, by limit theorems of sequences, we conclude:

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} \frac{\sin(s_n)}{s_n} = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} \sin s_n\right) = 0\left(\lim_{n\to\infty} \sin s_n\right) = 0.$$

Because $\frac{\sin(-x)}{(-x)} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$, we have the negative direction $(x \to -\infty)$ symmetrically for free.

For $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$, Example 9 of §19 precisely gives this result via the extension:

$$\tilde{f}(x) := \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

and defers the proofs of continuity of $\tilde{f}$ at 0, merely stating that this result reflects the fact that $\sin x$ is differentiable at 0 and its derivative is $\cos(0) = 1$. Because we do not define $\sin x$ in Ross, this suffices for our purposes.

□

Problem 20.12.
1. Sketch the function $f(x) = (x-1)^{-1}(x-2)^{-2}$.
2. Determine $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
3. Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to 1} f(x)$ if they exist.

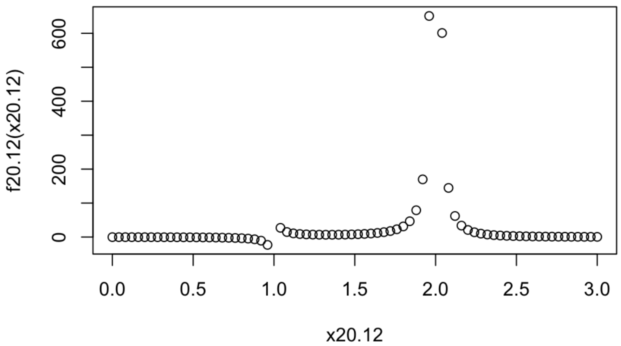

Figure 3: Sketch of 1/[(x-1)(x-2)]

Solution. Due to our general definition 20.1 given on the first page, we claim:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 1^+} f(x) = +\infty$$
$$\lim_{x\to 1^-} f(x) = -\infty$$

Hence because $\lim_{x\to 2^+} f(x) = \lim_{x\to 1^+} f(x) = +\infty$, we have

$$\lim_{x\to 2} f(x) = +\infty,$$

whereas because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we have that

$$\lim_{x\to 1} f(x) \text{ does not exist.}$$

For justification, we bash using Ross 20.4, properties of limits of functions (addition, multiplication and division). Working from left to right, we have:

$$\lim_{x\to 1^-}(x-1)^{-1}(x-2)^{-2} = \left(\lim_{x\to 1^-}(x-1)^{-1}\right)\left(\lim_{x\to 1^-}(x-2)^{-2}\right) = \left(\lim_{x\to 1^-}(x-1)^{-1}\right)(-1)^2 = -\infty$$
$$\lim_{x\to 1^+}(x-1)^{-1}(x-2)^{-2} = \left(\lim_{x\to 1^+}(x-1)^{-1}\right)\left(\lim_{x\to 1^+}(x-2)^{-2}\right) = \left(\lim_{x\to 1^+}(x-1)^{-1}\right)(-1)^2 = +\infty$$
$$\lim_{x\to 2^-}(x-1)^{-1}(x-2)^{-2} = \left(\lim_{x\to 2^-}(x-1)^{-1}\right)\left(\lim_{x\to 2^-}(x-2)^{-2}\right) = \left(\frac{1}{2}\right)\left(\lim_{x\to 2^-}(x-2)^{-2}\right) = +\infty$$
$$\lim_{x\to 2^+}(x-1)^{-1}(x-2)^{-2} = \left(\lim_{x\to 2^+}(x-1)^{-1}\right)\left(\lim_{x\to 2^+}(x-2)^{-2}\right) = \left(\frac{1}{2}\right)\left(\lim_{x\to 2^+}(x-2)^{-2}\right) = +\infty$$

□

Definition: Two-sided, Left-hand, Right-hand Limits -

(a) For $a \in \mathbb{R}$ and a function f, we write $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^s} f(x) = L$ for some set $S = J \setminus \{a\}$ where J is an open interval containing a. The limit $\lim_{x\to a} f(x)$ is called the **two-sided limit** of f at a. Note f need not be defined at a; and, even if f is defined at a, the value $f(a)$ need not equal $\lim_{x\to a} f(x)$. In fact, $f(a) = \lim_{x\to a} f(x)$ if and only if f is defined on an open interval containing a and f is continuous at a.

(c) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^-} f(x) = L$, provided $\lim_{x\to a^s} f(x) = L$ for some open interval $S = (c, a)$. The limit $\lim_{x\to a^-} f(x)$ is the **left-handed limit** of f at a.

(b) For $a \in \mathbb{R}$ and a function f, we write $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^s} f(x) = L$ for some open interval $S = (a, b)$. The limit $\lim_{x\to a^+} f(x)$ is the **right-hand limit** of f at a. Again, f need not be defined at a.

(d) For a function f, we write $\lim_{x\to\infty} f(x) = L$, provided $\lim_{x\to\infty^s} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, we write $\lim_{x\to-\infty} f(x) = L$ provided $\lim_{x\to-\infty^s} f(x) = L$ for some interval $S = (-\infty, b)$.

These limits defined here are unique; invariant on the exact choice of the set S.

Problem 20.19. The limits defined in Definition 20.3 do not depend on the choice of the set S. As an example, consider $a < b_1 < b_2$ and suppose f is defined on (a, b_2).

Show that if the limit $\lim_{x\to a^S} f(x)$ exists for either $S = (a, b_1)$ or $S = (a, b_2)$, then the limit exists for the other choice of S and these limits are identical. Their common value is what we write as $\lim_{x\to a^+} f(x)$.

Solution. We wish to prove the claim in the above definition, where the limits are unique, invariant on the exact choice of the set S. Let $a, b_1, b_2 \in \mathbb{R}$ with $a < b_1 < b_2$, and f is a function defined on (a, b_2).

First suppose $L_1 := \lim_{x\to a^S} f(x)$ exists for $S := (a, b_1)$. Define a sequence $s_n \in \hat{S} := (a, b_2)$ so that $s_n \to a$. We will show $\lim_{x\to a^S} f(x) = L_1 = \lim_{x\to a^{\hat{S}}}$. Because $a < \xi$ for all $\xi \in (a, b_1)$, convergence of $s_n \to a$ gives some N so that $(n \geq N) \implies s_n < b_1 < b_2$. Define a new sequence, $\hat{s}_n$ as the tail-end of s_n, starting with the index $N + 1$, so that for all valid indices n, $\hat{s}_n \in (a, b_1) = S$. Precisely, $\hat{s}_n$ take on indices $N + 1, N + 2, \ldots$. Because these two sequences are arbitrary within their definitions, we conclude

$$\lim_{x\to a^{\hat{S}}} f(x) = \lim_{x\to a^S} f(x) = L_1.$$

Conversely, now we suppose $L_2 := \lim_{x\to a^S} f(x)$ exists for $S := (a, b_2)$. Define a sequence $s_n \in \hat{S} := (a, b_1)$ so that $s_n \to a$. Because $(a, b_1) \subset (a, b_2)$, $s_n \in S = (a, b_2)$, with the same limit a. Hence

$$\lim_{x\to a^{\hat{S}}} f(x) = \lim_{x\to a^S} = L_2.$$

These limits L_1, L_2 are equivalent as we have shown equality in both directions, and of course, we write this as $\lim_{x\to a^+} f(x)$. □

Problem 21.10. Show there exist continuous functions

1. Mapping $(0,1)$ onto $[0,1]$,
2. Mapping $(0,1)$ onto $\mathbb{R}$,
3. Mapping $[0,1] \cup [2,3]$ onto $[0,1]$.
4. Explain why there are no continuous functions mapping $[0,1]$ onto $(0,1)$ or $\mathbb{R}$.

Solution. To show existence of continuous functions that perform the desired mappings, it suffices to simply construct an example for each.

(1) Consider the function $f(x) := |\sin(42x)|$, with $\text{dom}(f) := (0,1)$. By continuity of $\sin(x)$ and the scaling factor, and because we know $-1 \leq \sin(x) \leq 1$, we conclude that $(0,1) \overset{f}{\mapsto} [0,1]$, as desired. If the reader is unconvinced f continuous implies $|f|$ continuous, we refer to the general direction of Ross, where this result was likely explicitly proven. Otherwise, we know that compositions of functions are continuous, and we know already that $g(x) := |x|$ is continuous; hence $g(\sin(42x))$ is continuous.

(2) Consider the function $g(x) := \tan\left(\pi x - \frac{\pi}{2}\right)$, defined with $\text{dom}(g) := (0,1)$ (and we verify this is its canonical domain). Then just as we accept on faith (previous homework) that $\sin(x)$ and $\cos(x)$ are continuous on $\mathbb{R}$, because $\forall_{x \in (0,1)} \cos(x) \neq 0$, we dare to define $\tan(x) := \frac{\sin(x)}{\cos(x)}$ and proclaim it is continuous. Hence this proves existence of a continuous function mapping $(0,1) \overset{g}{\mapsto} \mathbb{R}$, as desired.

(3) Simply define the function

$$h(x) := \begin{cases} x, & x \in [0,1] \\ 3 - x, & x \in [2,3] \end{cases}$$

with $\text{dom}(h) := [0,1]$. As this is a piecewise polynomial defined on two disjoint intervals, we conclude $h(x)$ is continuous. □

Problem 22.1. Show there do not exist continuous functions

1. Mapping $[0,1]$ onto $[0,1] \cup [2,3]$,
2. Mapping $(0,1)$ onto $\mathbb{Q}$.

Solution. (1) By the IVT, we have that if f is continuous on $[0,1]$, then f has the intermediate value property, which is precisely violated by the fact the proposed codomain $[0,1] \cup [2,3]$ is disjoint (not connected). Hence there cannot be such continuous f with $[0,1] \overset{f}{\mapsto} [0,1] \cup [2,3]$.

(2) Similarly by the IVT, if f is continuous on $(0,1)$, then f has the intermediate value property. The proposed codomain $\mathbb{Q}$ is dense; however, a previous homework exercise showed that between any two rational numbers lives an irrational number. We conclude $\mathbb{Q}$ is not connected, whereas $(0,1)$ is connected, so there cannot exist such continuous f with $(0,1) \overset{f}{\mapsto} \mathbb{Q}$.
If additional justification is desired, consider the theorem given by Ross 22.2 and its Corollary 22.3. □

Problem 22.5. Let E and F be connected sets in some metric space.

1. Prove that if $E \cap F \neq \emptyset$, then $E \cup F$ is connected.
2. Give an example to show $E \cap F$ need not be connected. Incidentally, the empty set **is** connected.

Solution. (1) Suppose E, F are collected sets in a metric space (X, d), with $E \cap F \neq \{\}$ given. We show $E \cup F$ is connected. Suppose for contradiction that $E \cup F$ is disconnected by some open sets $U, V \subset X$. Take $\xi_1 \in E \cap F$ (we are guaranteed such an element by nonemptiness). WLOG let $\xi_1 \in U$ as opposed to V. Then take $\xi_2 \in V \cap (E \cup F)$, so that WLOG $\xi_2 \in E$ (as opposed to F). However, such elements ξ_1, ξ_2 make U, V nonempty, which implies E is disconnected, a direct contradiction to our given hypothesis that E, F are connected in X.
(2) The problem states that the empty set is trivially connected, as it cannot be written as a union of non-empty disjoint open sets. Consider two connected sets $E, F \in \mathbb{R}^2$, given by

$$E := \{(x, x^2) | x \in \mathbb{R}\}$$
$$F := \{(x, 1) | x \in \mathbb{R}\},$$

where E is the canonical parabola in $\mathbb{R}^2$ and F is the constant function 1. Their intersection can be explicitly writen as:

$$E \cap F = \{(-1, 1), (1, 1)\} \subset \mathbb{R}^2,$$

and hence we conclude $E \cap F$ is not connected, although E, F are connected. $\square$