Math 104: Homework 6

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

17.10 Prove the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the $\varepsilon - \delta$ property in Theorem 17.2.

(a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;

Solution: Consider the sequence $x_n = 1/n$. Clearly $x_n \to 0$ and since $x_n > 0$ for each n we have $f(x_n) = 1$, so $\lim f(x_n) = 1$. However, $f(0) = 0 \neq 1$, so f is discontinuous at x_0.

(b) $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$, $x_0 = 0$

Solution: Consider the sequence $x_n = \frac{1}{\pi/2+2\pi n}$. Clearly $x_n \to 0$, but for each n we have $g(x_n) = \sin(\pi/2 + 2\pi n) = 1$, so $\lim g(x_n) = 1 \neq g(0) = 0$, meaning g is discontinuous at 0.

17.12 (a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

Solution: Let $x \in (a, b)$. Construct a sequence (r_n) as follows: for each n, choose r_n to be a rational number in the interval $(x - 1/n, x + 1/n)$. This is possible since the rational numbers are dense in $\mathbb{R}$. Then we must have $r_n \to x$ because $|r_n - x| < 1/n$ for each n. So by the assumption that f is continuous we must have that $f(r_n) \to f(x)$. On the other hand, we have that $f(r_n) = 0$ for each n because the r_n are all rational. So $f(x) = \lim f(r_n) = 0$, as desired.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$. *Hint*: Use part (a).

Solution: Define the function h by $h(x) = f(x) - g(x)$ for $x \in (a, b)$. Then we see that $h(r) = 0$ for every rational $r \in (a, b)$, so by the previous part $h(x) = 0$ identically on (a, b). This means $f(x) - g(x) = 0$, or equivalently $f(x) = g(x)$, for all $x \in (a, b)$.

18.5 (a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

Solution: Clearly if $f(a) = g(a)$ or $f(b) = g(b)$ then we are done, so we can assume $f(a) > g(a)$ and $f(b) < g(b)$. Then define h by $h(x) = f(x) - g(x)$ for $x \in [a, b]$. We see that $h(a) > 0$ and $h(b) < 0$, so by the Intermediate Value Theorem there is some $x_0 \in (a, b)$ for which $h(x_0) = 0$. Then at this x_0 we have $f(x_0) = g(x_0)$, as desired.

(b) Show Example 1 can be viewed as a special case of part (a).

Solution: Example 1 is an application of part (a) with g the function $g(x) = x$.

18.6 Prove $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

Solution: Let $f(x) = \cos x - x$ for $x \in [0, \pi/2]$. Then we see $f(0) = 1$ and $f(\pi/2) = -\pi/2$, so by the Intermediate Value Theorem there exists some $x \in (0, \pi/2)$ for which $f(x) = 0$. Then this x satisfies $\cos x = x$.

19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) $x^{17}\sin x - e^x \cos 3x$ on $[0,\pi]$

Solution: The given function is uniformly continuous on $[0,\pi]$ because $[0,\pi]$ is a closed interval (i.e. by Theorem 19.2).

(c) $f(x) = x^3$ on $[0,1]$

Solution: I meant to write $(0,1)$ here, but it's fine if you just said the function is continuous because $[0,1]$ is a closed interval. To make the argument for $(0,1)$ you can just say that since the function is continuous on $[0,1]$ it is uniformly continuous on $[0,1]$ by 19.2, so it is uniformly continuous on $(0,1)$ because $(0,1) \subseteq [0,1]$.

(d) $f(x) = x^3$ on $\mathbb{R}$

Solution: The given function is *not* uniformly continuous on $\mathbb{R}$. To prove this, take $\varepsilon = 1$ and let $\delta > 0$. We wish to find two points x and y with $|x-y| < \delta$ for which $|x^3 - y^3| > 1$. We can employ the same technique as in §19 Example 3 (p. 142) and fix our attention on pairs x, y where $y = x + \delta/2$. Then we have

$$\begin{aligned} \left|x^3 - y^3\right| = |x-y|\left|x^2 + xy + y^2\right| &= \frac{\delta}{2}\left|x^2 + x\left(x + \frac{\delta}{2}\right) + \left(x + \frac{\delta}{2}\right)^2\right| \\ &= \frac{\delta}{2}\left|3x^2 + \frac{3\delta}{2}x + \frac{\delta^2}{4}\right|. \end{aligned}$$

Now, by inspecting the graph of f, we can notice that, since it is (up to a sign) symmetric about the y-axis, we can further restrict our attention to positive values of x – that is, if it is possible to find the desired pair x, y at all, it will certainly be possible to find such a pair where $x > 0$. In this case, we see that the second factor in the last line above is bounded below by $3x^2$, so it suffices to choose x for which $(\delta/2)(3x^2) > 1$, meaning $x^2 > 2/(3\delta)$, so take $x > \sqrt{2/(3\delta)}$. Then by the above reasoning we have

$$\left|f(x) - f\left(x + \frac{\delta}{2}\right)\right| = \frac{\delta}{2}\left(3x^2 + \frac{3\delta}{2}x + \frac{\delta^2}{4}\right) > \frac{\delta}{2}3x^2 > 1,$$

as desired.

(e) $f(x) = \frac{1}{x^3}$ on $(0,1]$

Solution: We claim the given function is not uniformly continuous on $(0,1]$. To see this, we argue by contradiction. Suppose f were uniformly continuous on $(0,1]$. Then by Theorem 19.5 there would be a continuous extension $\tilde{f}$ on $[0,1]$. But then since $\tilde{f}$ is continuous on a closed interval, the we would have $\tilde{f}([0,1])$ is bounded, which is clearly not true because the original function f is unbounded on $(0,1]$ (to see this explicitly, we can make $\tilde{f}$ achieve the values n^3 for $n \in \mathbb{N}$ by taking $x = 1/n \in [0,1]$). So f could not have been uniformly continuous.

(f) $f(x) = \sin\frac{1}{x^2}$ on $(0,1]$

Solution: We claim the given function is not uniformly continuous on $(0,1]$. To see this, consider the sequence (x_n) defined by $x_n = (\pi/2 + n\pi)^{-1/2}$ for $n \in \mathbb{N}$. Then we see that

$$f(x_n) = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n.$$

Now, note that (x_n) converges to 0 and is hence Cauchy, but on the other hand $(f(x_n))$ does not converge and hence cannot be Cauchy. By Theorem 19.4, if f were uniformly continuous, we would have to have that $(f(x_n))$ is Cauchy when (x_n) is Cauchy, so the fact that it $(f(x_n))$ isn't Cauchy means that f cannot be uniformly continuous on the given set.

(g) $f(x) = x^2 \sin \frac{1}{x}$ on $(0, 1]$

Solution: We claim f is uniformly continuous. To see this, we define a function

$$\tilde{f}(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

We claim $\tilde{f}$ is a continuous extension of f to $[0, 1]$. To see this, we note that by the various theorems 17.4-17.5 we have that f, and hence $\tilde{f}$, is continuous at any point of $(0, 1]$, so we need only argue that $\tilde{f}$ is continuous at $x = 0$. For this, let $\varepsilon > 0$. Then take $\delta < \sqrt{\varepsilon}$. Then when $|x| < \delta$ we have

$$|\tilde{f}(x) - 0| = |x|^2 \left| \sin \left(\frac{1}{x} \right) \right| \leq |x|^2 < \delta^2 < \varepsilon,$$

so indeed $\tilde{f}$ is continuous at 0. By Theorem 19.5, since f can be extended continuously to $[0, 1]$ it is uniformly continuous.

19.2 (b) Prove the function $f(x) = x^2$ is uniformly continuous on the set $[0, 3]$ by directly verifying the $\varepsilon - \delta$ property in Definition 19.1.

Solution: Let $\varepsilon > 0$. We wish to find $\delta > 0$ so that whenever $x, y \in [0, 3]$ with $|x - y| < \delta$ we have $|x^2 - y^2| < \varepsilon$. Note that

$$|x^2 - y^2| = |x - y||x + y| \leq |x - y| (|x| + |y|) \leq |x - y|(3 + 3) = 6|x - y|,$$

so choose $\delta < \varepsilon/6$. Then when $x, y \in [0, 3]$ and $|x - y| < \delta$

$$|f(x) - f(y)| = |x - y||x + y| < \delta(|x| + |y|) \leq 6\delta < \varepsilon,$$

as desired.

19.6 (a) Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0, 1]$ but f is nevertheless uniformly continuous on $(0, 1]$. Compare with Theorem 19.6.

Solution: We have $f'(x) = x^{-1/2}/2$ for $x \in (0, 1]$. At values $x = 1/n$ we see $f'(x) = \frac{\sqrt{n}}{2}$ which assumes arbitrarily large values for $n \in \mathbb{N}$, so indeed f' is unbounded on $(0, 1]$. To see that f is uniformly continuous on $(0, 1]$, we define $\tilde{f}(x) = \sqrt{x}$ for $x \in [0, 1]$. Then clearly $\tilde{f}$ is a continuous extension of f, so f is uniformly continuous by Theorem 19.5.

(b) Show f is uniformly continuous on $[1, \infty)$.

Solution: We see that $f'(x) = x^{-1/2}/2$ is bounded on $[1, \infty)$, since it is bounded above by $1/2$ and below by 0. So by 19.6 f is uniformly continuous on this interval.

19.8 (a) Use the Mean Value Theorem to prove

$$|\sin x - \sin y| \leq |x - y|$$

for all x, y in $\mathbb{R}$; see the proof of Theorem 19.6.

Solution: Let $x, y \in \mathbb{R}$. The inequality certainly holds if $x = y$ since both sides become 0, so we assume without loss of generality that $x < y$. Then (taking for granted that sine is differentiable on the interval (x, y) and its derivative is cosine) the Mean Value Theorem tells us there is some $z \in (x, y)$ with

$$\frac{\sin y - \sin x}{y - x} = \cos z,$$

meaning $|\sin x - \sin y| = |x - y||\cos z|$ for some $z \in (x, y)$. Since $|\cos z| \leq 1$ this shows $|\sin x - \sin y| \leq |x - y|$, as desired.

(b) Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

Solution: Let $\varepsilon > 0$. Then take $\delta = \varepsilon$. Then when $x, y \in \mathbb{R}$ and $|x - y| < \delta$ we have by part (a) that

$$|\sin x - \sin y| \leq |x - y| < \delta = \varepsilon,$$

showing $\sin x$ is uniformly continuous on $\mathbb{R}$.