Math 104, Summer 2019

PSET #6 (due Monday 7/22/2019)

Definition 17.1: Continuous -
Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is **continuous at** x_0 in $\text{dom}(f)$ if, for every sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \text{dom}(f)$, then we say f is **continuous on** S. We say the function f is **continuous** if it is continuous on $\text{dom}(f)$.

Theorem 0.1. Ross 17.2. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in $\text{dom}(f)$ if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$(x \in \text{dom}(f), |x - x_0| < \delta) \implies |f(x) - f(x_0)| < \varepsilon.$$

Problem 17.10. Prove the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the $\varepsilon - \delta$ property in Theorem 17.2.

1. $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;
2. $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$, $x_0 = 0$

Solution. (1) We defined:

$$f(x) := \begin{cases} 1, & 0 < x \\ 0, & x \leq 0. \end{cases}$$

To show f is discontinuous at $x_0 = 0$, we simply use Ross 17.1 Definition of a continuous function and find a sequence $(x_n) \to 0$ but $f(x_n) \not\to f(0) = 0$. Simply consider $x_n := \frac{1}{n}$. Surely, $x_n \to 0$ while $f(x_n) \to 1$. However, $f(0) = 0 \neq 1$, so we conclude f is discontinuous at $x = 0$.

(2) We defined:

$$g(x) := \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

To show g is discontinuous at $x = 0$, define the sequence $x_n := \frac{1}{2\pi n + \frac{\pi}{2}}$. Then surely, $x_n \to 0$, but notice $\lim \sin\left(\frac{1}{x_n}\right) = 1 \neq 0$, so we conclude $g(x) = \sin\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$. □

Problem 17.12. 1. Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.
2. Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$. *Hint*: Use part (a).

Solution. (1) Fix arbitrary $x \in (a, b)$, and let $x_n \to x$ for any sequence x_n in (a, b). By denseness of $\mathbb{Q}$, there exist some $r \in (a, x)$ and $r' \in (x, b)$. In particular, the denseness property of $\mathbb{Q}$ gives the existence of a sequence of rational numbers $r_n \to x$, where denseness of $\mathbb{Q}$ allows us to assert that $r_n \in (a, b)$ for all n. Because $r_n \in \mathbb{Q}$, for all n, $f(r_n) = 0$.
However, by continuity of f (given in hypothesis), all sequences $x_n \to x$ in (a, b) must have the same limit $\lim f(x_n)$. Because $\lim f(r_n) = 0$, we must have $\lim f(x_n) = 0$, for any sequence x_n in (a, b). Recall we fixed x at the start. Continuity of f gives $f(x) = \lim f(x_n) = 0$, as desired. $\square$

Solution. Define $h := f - g$. Because f, g continuous on (a, b), continuity theorems give that h is continuous on (a, b). Notice that for all rational numbers r, our hypotheses give $f(r) = g(r)$, and hence for all such r, we have $h(r) = f(r) - g(r) = 0$. That is, for all rational $r \in (a, b)$, we have $h(r) = 0$, and h continuous on (a, b), precisely satisfying the hypotheses for part (1) above, which gives the result that $h(x) = 0$ for all real $x \in (a, b)$. Because $h(x)$ is identically 0 for all of $x \in (a, b)$, we conclude that for $x \in (a, b)$, $h(x) := f(x) - g(x) = 0 \implies f(x) = g(x)$, as desired. $\square$

Theorem 0.2. Ross 18.2: Intermediate Value Theorem.
If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$, and $y \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\})$, there exists at least one $x \in (a, b)$ such that $f(x) = y$.

Problem 18.5. 1. Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.
2. Show Example 1 can be viewed as a special case of part (a).

Example 1 in Ross page 135 is: 'Let f be a continuous function mapping $[0, 1]$ into $[0, 1]$. In other words, $\text{dom}(f) = [0, 1]$ and $f(x) \in [0, 1]$ for all $x \in [0, 1]$. Show f has a **fixed point**; i.e. a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$; we say x_0 is left fixed by f.'

Solution. Define $h := f - g$, so that by continuity theorems, h is also continuous on $[a, b]$. Further, our hypotheses imply $h(a) \geq 0$ and $h(b) \leq 0$. To show $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$, it suffices to show $h(x_0) = 0$ for some $x_0 \in [a, b]$.
We split into cases. First, (trivially) if $h(a) = 0$, take $x_0 := a$ and we are done. Similarly, if $h(b) = 0$, take $x_0 := b$, and we are done.
Henceforth, suppose $h(a) > 0$ and $h(b) < 0$, where we handled the equality cases above. Surely, $0 \in (h(b), h(a))$; hence by the IVT, we precisely have the existence of at least one $x \in (a, b)$ where $h(x) = 0$. $\square$

Solution. Notice that Example 1 (fixed point) follows from our above, simply by settin $g(x) := x$. That is, with $f(a) \geq a$ and $f(b) \leq b$ (we call this Invariance), we proved that we have $f(x_0) = g(x_0) := x_0$ for at least one $x_0 \in [a, b]$. Hence we have at least one fixed point on the interval $[a, b]$. $\square$

Problem 18.6. Prove $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

Solution. We wish to show $\cos(x)$ has a fixed point within the open interval $\left(0, \frac{\pi}{2}\right)$. Notice that because we are given an open interval, we cannot immediately invoke our result from 18.5. Consider the closed interval $[\frac{\pi}{6}, \frac{\pi}{3}] \subset (0, \frac{\pi}{2})$. Define $f(x) := \cos(x)$, and $g(x) := x$, and notice

$$f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \approx 0.866 > 0.5236 \approx g(\frac{\pi}{6}) = \frac{\pi}{6} = \frac{\pi}{6}$$

as well as

$$f(\frac{\pi}{3}) = \frac{1}{2} < 1.047 \approx g(\frac{\pi}{3}) = \frac{\pi}{3}.$$

Hence by exercise 18.5 (previous page), we have $x = \cos x$ for some $x \in [\frac{\pi}{6}, \frac{\pi}{3}] \in (0, \frac{\pi}{2})$, as desired.

□

Definition: Uniform Continuity -
Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous on** S if, for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$(x, y \in S, \qquad |x - y| < \delta) \implies |f(x) - f(y)| < \epsilon.$$

Theorem 0.3. Ross 19.2: If f is continuous on a closed interval $[a, b]$, then f is **uniformly continuous** on $[a, b]$.

Theorem 0.4. Ross 19.4: If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.

Theorem 0.5. Ross 19.5: A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

Problem 19.1. Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) $x^{17} \sin x - e^x \cos 3x$ on $[0, \pi]$

Solution. Define $f(x) := x^{17} \sin(x) - e^x \cos(3x)$. Because $x^{17}, \sin x, e^x, \cos x$ are continuous on $[0, \pi]$, we surely have $\cos(3x)$ continuous on $\left[0, \frac{\pi}{3}\right]$, and hence by continuity theorems, we conclude $f(x)$ is continuous on $[a, b]$. By Ross 19.2, we conclude that f is uniformly continuous on $[a, b]$. □

(c) $f(x) = x^3$ on $[0, 1]$

Solution. Because $f(x) := x^3$ is continuous on the closed interval $[0, 1]$, by Ross 19.2, we have that f is uniformly continuous on $[0, 1]$. □

(d) $f(x) = x^3$ on $\mathbb{R}$

Solution. Now $f(x) := x^3$ defined on all of $\mathbb{R}$, we claim f is not uniformly continuous on $\mathbb{R}$. To see this, we show the negation of the definition.
Let $\epsilon := 1 > 0$ and fix arbitrary $\delta > 0$. Take $y := x + \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$. However,

$$\begin{aligned} |f(x) - f(y)| &= \left| x^3 - \left(x + \frac{\delta}{2} \right)^3 \right| \\ &= \left| 3x^3 \frac{\delta}{2} + 3x \left(\frac{\delta}{2} \right)^2 + \left(\frac{\delta}{2} \right)^3 \right|, \end{aligned}$$

and because we are exhibiting an example where the definition of uniform continuity fails, we can take $x > 0$. Then we have the bound:

$$\begin{aligned} |f(x) - f(y)| = \left| 3x^3 \frac{\delta}{2} + 3x \left(\frac{\delta}{2} \right)^2 + \left(\frac{\delta}{2} \right)^3 \right| &< \left| \frac{3x^3\delta}{2} \right| \\ &= 1 \quad (\text{when } x = \sqrt{\frac{2}{3\delta}}), \end{aligned}$$

and so we take $x = \sqrt{\frac{2}{3\delta}}$ and $y = x + \frac{\delta}{2}$, which results in $|f(x) - f(y)| = 1 \not< \epsilon$, and we conclude $f(x)$ is not uniformly continuous on all $\mathbb{R}$. □

(e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$

Solution. Consider the sequence $s_n := \frac{1}{n}$. Because $s_n \to 0$, s_n is Cauchy. However, notice $f(s_n) = s_n^3$, with $f(s_n) \to \infty$, hence $f(s_n)$ is not Cauchy (does not converge). Ross 19.4 gives that if f is uniformly continuous on $(0, 1]$ and s_n is cauchy in $(0, 1]$, then the sequence $f(s_n)$ must also be Cauchy; hence we conclude f cannot be uniformly continuous on $(0, 1]$. □

(f) $f(x) = \sin \frac{1}{x^2}$ on $(0, 1]$

Solution. Define $s_n := \sqrt{\frac{1}{n\pi + \frac{\pi}{2}}}$, where $s_n \to 0$ (and hence is Cauchy. However, $f(s_n) = \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n$, which does not converge and thus is not Cauchy. By the theorem (Ross 19.4) that states that if f is uniformly continuous on $(0, 1]$ and s_n is Cauchy in $(0, 1]$ then $f(s_n)$ is Cauchy, because $f(s_n)$ is not Cauchy, we conclude that f cannot be uniformly continuous on $(0, 1]$. □

Theorem 0.6. Ross 19.6: Let f be a continuous function on an interval I (which may be bounded or unbounded). Let I^o be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I^o and if f' is bounded on I^o, then f is uniformly continuous on I.

(g) $f(x) = x^2 \sin \frac{1}{x}$ on $(0, 1]$

Solution. Because we are explicitly allowed to use any theorems we wish, we use Ross 19.6, whose claim (above) is proven via the MVT (haha great textbook). Nevertheless, we first show an extension $\tilde{f}$ is uniformly continuous on $[0, 1]$, which then by (a corollary of) the theorem (Ross 19.5) that states a function f is uniformly continuous on $(0, 1]$ if and only if an extension $\tilde{f}$ is continuous on $[0, 1]$, we have our desired result.
Consider the extension:

$$\tilde{f} := \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Now for a bound of the derivative on $x \in I^o := (0, 1)$, consider:

$$\begin{aligned} \tilde{f} &= x^2 \sin\left(\frac{1}{x}\right) \\ |f'(x)| &= \left|2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)\right| \\ &= \left|2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \\ &\leq |2x| \left|\sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right| \\ &\leq 1 \cdot 1 + 1 = 2, \end{aligned}$$

and because $\tilde{f}$ is bounded on I^o by 2, Ross 19.6 gives that $\tilde{f}$ is uniformly continuous on $[0, 1]$, and hence by the Ross 19.5, we have f uniformly continuous on $(0, 1]$ as desired.

□

Problem 19.2. (b) Prove the function $f(x) = x^2$ is uniformly continuous on the set $[0, 3]$ by directly verifying the $\delta - \varepsilon$ property in Definition 19.1.

Solution. Let $\epsilon > 0$ and take $x, y \in [0, 3]$. Consider:

$$|f(x) - f(y)| = |x^2 - y^2| = \underbrace{|x + y|}_{\leq 3+3} |x - y| \leq 6|x - y|.$$

Let $|x - y| < \delta$, with $\delta := \frac{\epsilon}{6}$. Then:

$$|f(x) - f(y)| \leq 6|x - y| < 6\delta = \epsilon,$$

as required.

□

Problem 19.6.
1. Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0, 1]$ but f is nevertheless uniformly continuous on $(0, 1]$. Compare with Theorem 19.6.
2. Show f is uniformly continuous on $[1, \infty)$.

Solution. (1) Suppose we know what a derivative is. Then $f(x) = x^{1/2}$ and $f'(x) = \frac{1}{2}x^{-1/2}$. Take $s_n := \frac{1}{n}$ (and verify it is in $(0, 1]$). Notice $s_n \to 0$, but $f'(s_n) \to \infty$, hence f' is unbounded on $(0, 1]$. Notice that Ross 19.6 is only a one-way implication, and hence this does not violate our theorem.

Now we show that f is uniformly continuous on $(0, 1]$. Because f is continuous on the closed interval $[0, 1]$ (we proved this result before) for x^p continuous, $x \in \mathbb{R}$, $p > 0$, f is uniformly continuous on the subset $(0, 1]$, by the relevant theorem (Ross 19.2). □

Solution. (2) Now we show f is uniformly continuous on $[1, \infty)$. Take $x_1, x_2 \in [1, \infty)$. Notice that for all $\xi \in [1, \infty)$, $|f'(\xi)| \leq \frac{1}{2}$. To see this bound explicitly, notice that $x \geq 1$ implies $f'(x) > 0$ yet $f''(x) = \frac{-1}{4}x^{-3/2} < 0$. Then (by Ross 19.6 or) by the MVT and our bound for $|f'(\xi)|$, we satisfy the Lipschitz condition:

$$|f(x_1) - f(x_2)| \leq \frac{1}{2}|x_1 - x_2|,$$

and setting $\delta := \epsilon$, we easily have that $|x_1 - x_2| < \delta$ implies

$$|f(x_1) - f(x_2)| \leq \frac{1}{2}|x_1 - x_2| < \frac{\delta}{2} = \frac{\epsilon}{2} < \epsilon,$$

as required for uniform continuity. □

Problem 19.8.
1. Use the Mean Value Theorem to prove

$$|\sin x - \sin y| \leq |x - y|$$

for all x, y in $\mathbb{R}$; see the proof of Theorem 19.6.
2. Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

Solution. Define $f(x) := \sin(x)$, with $x, y \in \mathbb{R}$. By the MVT, there exists $\xi \in (x, y)$ such that $f'(\xi) = \frac{f(y)-f(x)}{y-x}$, so

$$|f(x) - f(y)| = |f'(\xi)||x - y|.$$

However, we know for all $\xi \in \mathbb{R}$, $|f'(\xi)| = |\cos \xi| \leq 1$. Hence we have a bound:

$$|f(x) - f(y)| \leq 1|x - y|,$$

precisely as desired. This is a basic example of the Lipschitz condition with $C = 1$, independent of $x, y \in \mathbb{R}$ (notice that a bounded first derivative implies Lipschitz). □

Solution. Let $\epsilon > 0$, and set $\delta := \epsilon$. Then for $x, y \in \mathbb{R}$, $|y - x| < \delta$ implies:

$$|\sin y - \sin x| \leq |y - x| < \delta = \epsilon,$$

as required to show $\sin x$ is uniformly continuous on $\mathbb{R}$. □