Math 104: Homework 5

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

15.1 Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{(-1)^n}{n}$
Solution: This series converges by the alternating series test.

(b) $\sum \frac{(-1)^n n!}{2^n}$
Solution: Note that we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2} \to \infty,$$

so the series diverges by the ratio test.

15.2 Repeat Exercise 15.1 for the following.

(a) $\sum [\sin(\frac{n\pi}{6})]^n$
Solution: For all terms where $n = 3 + 12k$ for some $k \in \mathbb{N}$ we see that $[\sin(\frac{n\pi}{6})]^n = 1$, so clearly the terms do not approach 0, meaning the series diverges.

(b) $\sum [\sin(\frac{n\pi}{7})]^n$
Solution: Note that the sequence $\sin(\frac{n\pi}{7})$ is periodic and we never have $\sin(\frac{n\pi}{7}) = \pm 1$. This means we must have

$$\left|\sin\left(\frac{n\pi}{7}\right)\right| < 1 \qquad \text{for} \qquad n = 1, 2, \ldots, 14,$$

so we can find $r < 1$ with

$$r > \max\left\{\sin\left(\frac{n\pi}{7}\right) \,:\, n = 1, 2, \ldots, 14\right\},$$

so that

$$\left|\sin\left(\frac{n\pi}{7}\right)\right|^n < r^n$$

for each n, which shows that the given series converges absolutely by comparison with the convergent geometric series $\sum r^n$, and hence the given series is convergent.

15.3 Show $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Solution: First suppose $p > 1$. Then note that we have

$$\begin{aligned}\int_2^n \frac{1}{x(\log x)^p}dx = \int_{\log 2}^{\log n} \frac{1}{u^p}du &= -\frac{1}{p-1}\left(\frac{1}{(\log n)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right) \\ &= \frac{1}{p-1}\left(\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log n)^{p-1}}\right).\end{aligned}$$

Letting $n \to \infty$ in the last expression we see that the improper integral $\int_2^\infty \frac{1}{x(\log x)^p}dx$ converges to $1/((p-1)(\log 2)^{p-1})$, so by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

Now suppose $p = 1$. Then in this case we have

$$\int_2^n \frac{1}{x \log x} = \int_{\log 2}^{\log n} \frac{1}{u} du = \log(\log n) - \log(\log 2),$$

and clearly the latter expression diverges to ∞ as $n \to \infty$, meaning the series diverges by the integral test.

Now let $0 < p < 1$. Then again we have

$$\int_2^n \frac{1}{x(\log x)^p} dx = -\frac{1}{p-1}\left(\frac{1}{(\log n)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right) = \frac{1}{1-p}\left((\log n)^{1-p} - (\log 2)^{1-p}\right),$$

and since $p < 1$ the expression on the right also diverges to ∞ as $n \to \infty$, so the series again diverges.

Lastly, we clearly have that for $p \leq 0$ the terms in the series do not converge to 0 so the series diverges in those cases as well.

15.6 (a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

Solution: If we take $a_n = 1/n$ we see $\sum a_n$ is the harmonic series which diverges but $\sum a_n^2 = \sum \frac{1}{n^2}$ converges, say by the previous exercise.

(b) Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.

Solution: If $a_n \geq 0$ for all n and $\sum a_n$ converges, then since $a_n \to 0$ we can find some N for which $a_n < 1$ for all $n \geq N$. Then note that clearly $\sum_{n=N}^{\infty} a_n$ converges and $0 \leq a_n^2 < a_n$ for $n \geq N$, so by the comparison test we have that $\sum_{n=N}^{\infty} a_n^2$ converges as well. Then we see

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2$$

converges, as desired. We could also have cited the previous exercise where we proved the more general fact $\sum a_n^p$ converges when $p > 1$.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Solution: The classic example of this is the series $\sum \frac{(-1)^n}{\sqrt{n}}$ (which converges by the alternating series test), for which we have that $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

17.1 Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

(a) Give the domains of $f+g$, fg, $f \circ g$ and $g \circ f$.

Solution: We have that $f+g$ and fg are defined precisely where f and g are each individually defined, i.e. on the intersection of their domains. The domain of $f(x)$ is $(-\infty, 4]$ and the domain of g is $\mathbb{R}$ so $f+g$ and fg both have domain $(-\infty, 4] \cap \mathbb{R} = (-\infty, 4]$.

The composition $f \circ g$ is defined precisely where g is defined and takes values within the domain of f. Since g is defined everywhere, this means the domain of $f \circ g$ is simply the set of $x \in \mathbb{R}$ where $g(x) \in (-\infty, 4]$, i.e. where $x^2 \leq 4$. This, in turn, is the interval $[-2, 2]$.

The composition $g \circ f$ is defined where f is defined and takes values in the domain of g. Since g is defined everywhere, this is just the domain of f, which is $(-\infty, 4]$.

(b) Find the values $f \circ g(0)$, $g \circ f(0)$, $f \circ g(1)$, $g \circ f(1)$, $f \circ g(2)$ and $g \circ f(2)$.

Solution: We have:

$$\begin{aligned} f \circ g(0) &= f(g(0)) = f(0) = 2, \\ g \circ f(0) &= g(f(0)) = g(2) = 4, \\ f \circ g(1) &= f(g(1)) = f(1) = \sqrt{3}, \\ g \circ f(1) &= g(f(1)) = g(\sqrt{3}) = 3, \\ f \circ g(2) &= f(g(2)) = f(4) = 0, \qquad \text{and} \\ g \circ f(2) &= g(f(2)) = g(\sqrt{2}) = 2. \end{aligned}$$

(c) Are the functions $f \circ g$ and $g \circ f$ equal?

Solution: No, since they take different values at $x = 0$ (for instance).

(d) Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

Solution: Sadly, $f \circ g(3)$ fails to be meaningful because $g(3) = 9$, and $f(9)$ is not defined. On the other hand, $g \circ f(3)$ is meaningful because f is defined at 3 and g is defined at $f(3) = 1$. In the latter case we have $g \circ f(3) = g(1) = 1$.

17.2 Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$.

(a) Determine the following functions: $f+g$, fg, $f \circ g$, $g \circ f$. Be sure to specify their domains.

Solution: The function $f + g$ is defined on all of $\mathbb{R}$ and we have

$$(f+g)(x) = \begin{cases} x^2 & x < 0 \\ x^2 + 4 & x \geq 0 \end{cases}$$

The function fg is defined on all of $\mathbb{R}$ and we have

$$(fg)(x) = \begin{cases} 0 & x < 0 \\ 4x^2 & x \geq 0 \end{cases}$$

The function $f \circ g$ is defined on all of $\mathbb{R}$ and since $g(x) \geq 0$ for all x we have $(f \circ g)(x) = 4$ for all $x \in \mathbb{R}$.

The function $g \circ f$ is defined on all of $\mathbb{R}$ and we have

$$(g \circ f)(x) = \begin{cases} 0 & x < 0 \\ 16 & x \geq 0 \end{cases}$$

(b) Which of the functions f, g, $f+g$, fg, $f \circ g$, $g \circ f$ is continuous?

Solution: The function $f + g$ is not continuous at $x = 0$ because for any $\delta > 0$ we can always find some x with $-\delta < x < 0$ so that $|f(x) - f(0)| = |0 - 4| = 4$.

The function fg is, in fact, continuous. Clearly it is continuous at points $x \neq 0$ because at such points it is identical to one of the two continuous functions 0 or $4x^2$ in a sufficiently small neighborhood of that point, so the only point in question is $x = 0$. However, at this point we see that for $\varepsilon > 0$ we can take $\delta < \sqrt{\varepsilon}/4$ and that will ensure that for $|x| < \delta$ we have $|f(x)| < \varepsilon$ because either $f(x) = 0$ or $f(x) = 4x^2$ and the inequality holds in either case.

The function $f \circ g$ is constant and therefore continuous.

The function $g \circ f$ is discontinuous by an argument almost identical to the one for $f + g$.

17.3 Accept on faith that the following familiar functions are continuous on their domains: $\sin x$, $\cos x$, e^x, 2^x, $\log_e x$ for $x > 0$, x^p for $x > 0$ [p any real number]. Use these facts and theorems in this section to prove the following functions are also continuous.

(b) $[\sin^2 x + \cos^6 x]^\pi$

Solution: Since $\sin x$ is continuous, we have by 17.4(ii) that $\sin^2 x$ is continuous. Similarly by several applications of 17.4(ii) we have that $\cos^6 x$ is continuous since $\cos x$ is continuous. So by 17.4(i) we have that $\sin^2 x + \cos^6 x$ is continuous. Then note that for all x we have that $\sin^2 x + \cos^6 x > 0$ (which is actually partly because $\sin x$ and $\cos x$ never simultaneously vanish) so by assumption $g(x) = x^\pi$ is continuous at all points in the image of $f(x) = \sin^2 x + \cos^6 x$ so by 17.5 we have that $g \circ f(x) = [\sin^2 x + \cos^6 x]^\pi$ is continuous everywhere.

(c) 2^{x^2}

Solution: we can certainly take for granted that the identity function $f(x) = x$ is continuous everywhere so by 17.4(ii) we have that x^2 is continuous everywhere. Then since 2^x is continuous everywhere we have by 17.5 that 2^{x^2} is continuous everywhere.

(e) $\tan x$ for $x \neq$ odd multiple of $\pi/2$

Solution: When $x \neq$ an odd multiple of $\pi/2$ we have that $\cos x$ is not zero. So by 17.4(iii) since $\sin x$ and $\cos x$ are continuous at such an x we have that $\sin x / \cos x = \tan x$ is continuous there as well.

(f) $x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$

Solution: When $x \neq 0$ we have by 17.4(iii) that $\frac{1}{x}$ is continuous because 1 and x are each continuous. So since $\sin x$ is defined and continuous everywhere we have by 17.5 that $\sin\left(\frac{1}{x}\right)$ is continuous at any $x \neq 0$. So since x is continuous again by 17.4(ii) we have that $x \sin\left(\frac{1}{x}\right)$ is continuous.

(h) $\frac{1}{x} \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$

Solution: As in the previous part, we have $1/x$ is continuous when $x \neq 0$, so by 17.4(ii) we have $1/x^2$ is also continuous when $x \neq 0$. Again since $\sin x$ is continuous everywhere by 17.5 we have $\sin\left(\frac{1}{x^2}\right)$ is continuous, and as we remarked before $1/x$ is continuous so by 17.4(ii) again $\frac{1}{x} \sin\left(\frac{1}{x^2}\right)$ is also continuous.