Math 104, Summer 2019

PSET #5 (due Thursday 7/18/2019)

Problem 15.1 - 15.2. Determine which of the following series converge. Justify your answers.

$$\sum \frac{(-1)^n}{n}; \qquad \sum \frac{(-1)^n n!}{2^n}; \qquad \sum \left[\sin\left(\frac{n\pi}{6}\right)\right]^n; \qquad \sum \left[\sin\left(\frac{n\pi}{7}\right)\right]^n$$

Solution. (1) $\sum \frac{(-1)^n}{n}$ converges. Define $a_n := \frac{(-1)^n}{n}$. To prove this formally, we write:

$$\sum \frac{(-1)^n}{n} = -\sum \frac{(-1)^{n+1}}{n},$$

and we observe that our hypotheses are met (alternating series with terms $a_n \to 0$) to invoke the Alternating Series Theorem and conclude that our original series converges. □

Solution. (2) $\sum \frac{(-1)^n n!}{2^n}$ does not converge. Define $a_n := \frac{n!}{2^n}$. By the ratio test, consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}} \frac{2^n}{(n!)},$$

where we drop the negative due to the absolute value. Further notice that $a_n \neq 0$ for all n, so use of the ratio test is valid.

□

Solution. (3) $\sum \left[\sin\left(\frac{n\pi}{6}\right)\right]^n$ does not converge.
Consider the set $U \subset \mathbb{N}^+$ of indices k for which $a_{6k-3} = 1$. Because $\left[\sin\left(\frac{3\pi}{6}\right)\right]^3 = 1^3 = 1$, we have $1 \in U$. Now assume $k \in U$, so that

$$\begin{aligned}
a_{6k+3} &= \left[\sin\left(\frac{(6k-3)\pi}{6}\right)\right]^{6k-3} \\
&= \left[\sin\left(k\pi - \frac{\pi}{2}\right)\right]^{6k+3} \\
&= \left[\sin(k\pi)\cos\left(\frac{-\pi}{2}\right) + \sin\left(\frac{-\pi}{2}\right)\cos(k\pi)\right]^{6k+3} \qquad \text{(trig identity)} \\
&= [\sin(k\pi)\cdot 0 - 1\cdot\cos(k\pi)]^{6k+3} \\
&= [-\cos(k\pi)]^{6k+3} = [(-1)^k]^{3k} = \begin{cases} 1, & k \text{ odd} \\ -1, & k \text{ even}, \end{cases}
\end{aligned}$$

and because we can find these terms valued at 1 and -1 at arbitrarily large indices n, we conclude the sequence $a_n \not\to \infty$, and hence the original series $\sum a_n$ does not converge. (Notice we don't say it diverges). □

Solution. (4) $\sum \left[\sin\left(\frac{n\pi}{7}\right)\right]^n$ converges.
Intuitively, notice that for all n, $\sin\left(\frac{n\pi}{7}\right) < 0.999$; to see why, simply consider that no n will give $\sin\left(\frac{k\pi}{2}\right)$, where k is some odd positive integer. Define $a_n := \sin\left(\frac{n\pi}{7}\right)$. Because

$$\sum |a_n| < \sum 0.999^n,$$

by comparison test to a convergent geometric series ($0.999 < 1$), we conclude our original series converges.

□

Problem 15.3. Show $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Solution. Define: $a_n := \frac{1}{n(\log n)^p}$.
$(\Longleftarrow)$ Suppose $p > 1$. Assuming this is logarithm base 10 (the result is the same if we treat log as ln here, as we begin indexing at 2), first notice that all terms of the series are strictly positive. By Lemma 0.1 below, $\sqrt{n} \geq \log(n)$ for $n \geq 101$. Then, consider:

$$\begin{aligned}\sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{100} a_n + \sum_{n=101}^{\infty} \frac{1}{n(\log n)^p} \\ &\leq \sum_{n=2}^{100} a_n + \sum_{n=101}^{\infty} \frac{1}{n}\left(\frac{1}{n}\right)^{p/2} \\ &= \sum_{n=2}^{100} a_n + \sum_{n=101}^{\infty} \frac{1}{n^q},\end{aligned}$$

where $q := 1 + \frac{p}{2}$. Because $p > 1$, we conclude $q > 1$, and by the p-test, we have that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges, as desired.
$(\Longrightarrow)$ Suppose $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges to some s. That is, $\sum_{n=2}^{\infty} a_n \to s$. For contradiction, suppose $p \leq 0$. Then our statement must hold for $p := 0$, which claims:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(\log n)^{|p|}}{n} &= \sum_{n=2}^{10} \frac{(\log n)^{|p|}}{n} + \sum_{n=11}^{\infty} \frac{(\log n)^{|p|}}{n} \\ &\leq \sum_{n=2}^{10} \frac{(\log n)^{|p|}}{n} + \sum_{n=11}^{\infty} \frac{1}{n}\end{aligned}$$

but we know (and have proven many times) that this harmonic series (starting from 11) diverges; a contradiction. So we must have $p > 0$. Now consider for contradiction that $0 < p \leq 1$, which similarly gives:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} = \sum_{n=2}^{10} \frac{1}{n\,(\log n)^p} + \sum_{n=11}^{\infty} \frac{1}{n\,(\log n)^p},$$

and I'm stuck on how to force this to diverge... I refuse to use the integral test as we haven't defined what an integral is. I'm sure there's a clever way to do this, but I haven't found it. Will ask in OH on Friday.

□

Lemma 0.1. Where log is defined with base 10, $\log(n) \leq \sqrt{n}$ for all $n \geq 101$.

Proof. Consider the subset $U \in \mathbb{N}$ of k for which $n \geq (100 + k)$, the above is true. That is, let U be the set of k for which

$$\log(100 + k) \leq \sqrt{100 + k}. \tag{1}$$

A quick calculator test gives $1 \in U$:

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> log(101, base=10) - sqrt(101)
[1] -8.045554
> log(10000000001, base=10) - sqrt(10000000001)
[1] -99990
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Now we assume $k \in U$, so that (1) holds. If $k \in U$ implies $k+1 \in U$, by induction we are done. Consider... that I found this actually surprisingly tricky and I have a small brain so let's put off proof of this lemma until we have Taylor expansions... or integrals defined so we can use the integral test... (pls have mercy). □

In office hours on Friday, we talked about using the concept of an integral and the integral test to gain some intuition about bounding the previous problem (15.3).
Consider:

$$\begin{aligned}\int_n^{n+1} \frac{1}{x(\log x)^p}\ dx &= \int_{\log n}^{\log(n+1)} u^{-p}\ du \\ &= \left(\frac{1}{1-p}u^{1-p}\right)_{\log n}^{\log(n+1)} \\ &= \frac{1}{1-p}\left[\log(n+1)^{1-p} - \log(n)^{1-p}\right]\end{aligned}$$

Problem 15.6.

1. Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
2. Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.
3. Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Solution. (1) The canonical example of such a sequence where $\sum a_n$ diverges but $\sum a_n^2$ converges is $a_n := \frac{1}{n}$. That is, by the p-series test, we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $p = 1 \not> 1$, because $a_n < 1$ for all $n \geq 2$. Also by the p-series test, because $p = 2 > 1$ and $a_n < 1 \implies a_n^2 < 1$ for all $n \geq 2$, we have that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (In fact, because it was mentioned in class, we know this sums to $\frac{\pi^2}{6}$.)

□

Solution. (2) Suppose $a_n \geq 0$ for all n, and that $\sum a_n$ converges. Notice this implies $a_n \to 0$, and hence there exists some M_1 for which $n > M_1$ implies $a_{n+1} \leq a_n$; that is, the tail-end of the sequence a_n must be nonincreasing, as the sequence converges to 0 but is strictly positive. Particularly, there exists some M_2 where all $n > M_2$ implies $a_{M_2} < 1$. Consider:

$$\sum a_n^2 = \sum_{n=1}^{M_2} a_n^2 + \sum_{n=M_2+1}^{\infty} a_n^2,$$

where the left term is a finite sum, and the right term is an infinite sum of terms all less than 1. However, notice that for all $n > M_2$, $a_n < 1$ and hence $a_n^2 < a_n < 1$ (strict inequality). Hence we have:

$$\begin{aligned} \sum a_n^2 &= \sum_{n=1}^{M_2} a_n^2 + \sum_{n=M_2+1}^{\infty} a_n^2 \\ &\leq \sum_{n=1}^{M_2} a_n^2 + \sum_{n=M_2+1}^{\infty} a_n, \end{aligned}$$

which we know is finite by hypothesis. Additionally by hypothesis, all terms are nonnegative, so we conclude that $\sum a_n$ convergent implies $\sum a_n^2$ convergent. □

This proof is much shorter if we invoke 14.7 with case-work.

Solution. (3) One such intuition here is that we can make $\sum a_n$ converge via alternating series, whereas $\sum a_n^2$ is guaranteed to be the sum of all nonnegative terms. Because a_n might not converge absolutely, then $\sum a_n^2$ may fail to converge. To see this in action, we know that $\sum \frac{1}{n}$ diverges (harmonic series or p-test with $p = 1$). Let:

$$a_n := \frac{(-1)^n}{n^k},$$

for some $0 < k \leq \frac{1}{2}$. By the alternating series test, we know $\sum a_n$ converges. If this begs the question, we explicitly note $a_n \to \infty$, and $a_n a_{n+1} < 0$. Now,

$$\sum a_n^2 = \sum \left(\frac{(-1)^n}{n^k} \right) = \sum \left(\frac{1}{n^{2k}} \right),$$

but $0 < 2k < 1$, and by the p-test, because all terms are strictly positive, $\sum a_n^2$ fails to converge (in fact, the p-test gives that it diverges), precisely as desired. If an actually specific example is required, set $k := \frac{1}{2}$ to get the same result. □

Problem 17.1. Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

1. Give the domains of $f+g$, fg, $f \circ g$ and $g \circ f$.
2. Find the values $f \circ g(0)$, $g \circ f(0)$, $f \circ g(1)$, $g \circ f(1)$, $f \circ g(2)$ and $g \circ f(2)$.
3. Are the functions $f \circ g$ and $g \circ f$ equal?
4. Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

Solution. (1) Notice $\text{Im}(f) = \text{Im}(g) = [0, \infty)$, as they are defined; however, their domains are different. Consider that $f(x)$ is only defined for $x \leq 4$, whereas $g(x)$ is defined for all $x \in \mathbb{R}$. Hence the domain of $f+g$ is restricted to all $x \leq 4$, which we write as $(-\infty, 4]$.

By the same token as above, the domain of $f \cdot g$ is all $x \leq 4$, which we write as $(-\infty, 4]$.

The domain of the composite function $f \circ g = f[g(x)]$ is restricted by f only defined on $x \leq 4$. Hence for $f \circ g$ to be defined, we need $g(x) \leq 4$, which imposes the restriction $-2 \leq x \leq 2$. We conclude the composite function $f \circ g$ is defined on the domain $[-2, 2]$.

The function g is defined on all $x \in \mathbb{R}$, so the composite function $g \circ f$ is defined for the same domain as $f(x)$, namely $x \leq 4$, which we write as $(-\infty, 4]$. □

Solution. 2. The desired values are:

$$
\begin{aligned}
f \circ g(0) &= 2 \\
g \circ f(0) &= 4 \\
f \circ g(1) &= \sqrt{3} \\
g \circ f(1) &= 3 \\
f \circ g(2) &= 0 \\
g \circ f(2) &= 2,
\end{aligned}
$$

which coincidentally looks strikingly like some Newton divided difference table. □

Solution. 3. No, $f \circ g \neq g \circ f$, as can be seen via part (1) above; their domains are different and thus they are easily different functions, without further consideration. □

Problem 17.2. Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$.

1. Determine the following functions: $f+g$, fg, $f \circ g$, $g \circ f$. Be sure to specify their domains.
2. Which of the functions f, g, $f+g$, fg, $f \circ g$, $g \circ f$ is continuous?

Solution. (1) Consider:

$$
\begin{aligned}
[f+g](x) &= \begin{cases} 4 + x^2, & 0 \leq x \\ x^2, & x < 0 \end{cases} \\
[fg](x) &= \begin{cases} 4x^2, & 0 \leq x \\ 0, & x < 0 \end{cases} \\
[f \circ g](x) &= f(x^2) = 4, \forall x \in \mathbb{R} \\
[g \circ f](x) &= \begin{cases} 16, & 0 \leq x \\ 0, & x < 0 \end{cases}
\end{aligned}
$$

(2) All but $g \circ f$ (where we have a 'jump discontinuity') are continuous. □

Problem 17.3. Accept on faith that the following familiar functions are continuous on their domains: $\sin x$, $\cos x$, e^x, 2^x, $\log_e x$ for $x > 0$, x^p for $x > 0$ [any $p \in \mathbb{R}$]. Use these facts and theorems in this section to prove the following functions are also continuous.
(b) $[\sin^2 x + \cos^6 x]^\pi$
(c) 2^{x^2}
(e) $\tan x$ for $x \neq$ odd multiple of $\pi/2$
(f) $x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$
(h) $\frac{1}{x} \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$

Solution. (b) We are given that the function x^p is continuous for any $p \in \mathbb{R}$, so it must be so for $p = 2, 6$. Further, we are given $\sin(x), \cos(x)$ are continuous. Invoking Ross 17.4 and 17.5, we have $\sin^2 x + \cos^6 x$ is continuous, noting that not only do even powers force nonnegative values but also that $\sin^2 x + \cos^6 x > 0$ for all $x \in \mathbb{R}$ (we can see this via contradiction or via a phaseshift, defining $\cos(x) := \sin(x + \frac{\pi}{2})$. Because $\sin^2 x + \cos^6 x > 0$ and $\pi \in \mathbb{R}$ and we have x^p continuous for $x > 0$, Ross 17.5 gives that $\left[\sin^2 x + \cos^6 x\right]^\pi$ is continuous, as desired.

(c) Assume this ambiguous expression is $2^{(x^2)}$ and not $(2^x)^2$, under the premise that otherwise the expression would be written simply as $2^{2x} = 4^x$. Then because $f(x) := x^2$ is continuous for $x > 0$ ($p = 2 \in \mathbb{R}$) and because $g(x) := 2^x$ is given to be continuous for $x > 0$, by invoking Ross 17.5, we have $g \circ f(x)$ is continuous, as desired. Similarly, $f \circ g$ is also continuous, so we can do away with our assumption to cover both cases of ambiguity.

(e) Because $x \neq$ an odd multiple of $\frac{\pi}{2}$, we have $\cos(x) \neq 0$. This allows us to familiarly let $\tan x = \frac{\sin x}{\cos x}$, each of which we are given to be continuous, and hence Ross 17.4 gives that $\tan x$ is continuous as desired (for our given domain $x \neq$ odd multiple of $\pi/2$).

(f) Assume we know $h(x) := x$ is continuous. (If this isn't permitted, because $p = 1 \in \mathbb{R}$, x is continuous for $x > 0$. Further, if we want $h(x)$ continuous for not only $x > 0$ but rather all of $x \neq 0$, define $h(-x) := -h(x) = -x$.) Notice that $f(x) := \frac{1}{x} = x^{-1}$, and because $-1 \in \mathbb{R}$, our given lets us conclude $f(x)$ is continuous for $x > 0$, which assures $x \neq 0$. Similarly, define $f(-x) := -f(x)$, so that we have f continuous for all $x \neq 0$. Further, we are given that $g(x) := \sin(x)$ is continuous, and by our assumption $h(x) = x$ continuous, invoking Ross 17.5, we conclude $h \circ g \circ f$ is continuous for $x > 0$. To get $x < 0$, we simply (ab)use our knowledge that h, g, f are all odd functions. If this is begging the question, define $f(-x) := -f(x), g(-x) := -g(x), h(-x) := -h(x)$ and verify these definitions are correct to get this result. If more is required, I should really re-think my life.

(h) Similar to in (f), define $f(x) := \frac{1}{x^2} = x^{-2}$, and because $-2 \in \mathbb{R}$, we have $f(x)$ continuous for $x > 0$. Define $f(-x) := -f(x)$, and we then have a new $f(x) = \frac{1}{x^2}$ continuous for $x \neq 0$. Because $g(x) := \sin x$ is continuous its entire domain, Ross gives that $g \circ f$ is continuous for all $x \neq 0$. Further, define $h(x) := \frac{1}{x}$, where $-1 \in \mathbb{R}$ gives that $h(x)$ is continuous for $x > 0$ by our hypothesis. Define $h(-x) := -h(x) = -\frac{1}{x}$, so that $h(x)$ is continuous for all $x \neq 0$. Then by Ross, the composite function $h \circ g \circ f$ is continuous for all $x \neq 0$. $\square$