Math 104: Homework 4

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

13.4 Prove (iii) and (iv) in Discussion 13.7 [page 87 in the textbook].

Solution: For (iii), we wish to show the union of any collection of open sets in S is open. So let $\mathcal{U}$ be an arbitrary family of open subsets of S. Then put

$$E = \bigcup\{U \,:\, U \in \mathcal{U}\}.$$

We wish to show E is open. So let $x \in E$. Then we must have $x \in U$ for some $U \in \mathcal{U}$. This means we can find $r > 0$ so that $B_r(x) := \{y \in S \,:\, d(x, y) < r\}$ is contained in U. But then this means $B_r(x) \subseteq E$ as well since E is a union of sets including U. So x is interior to E, and since $x \in E$ was arbitrary we see that E is indeed open.

For (iv), let $U_1, \dots, U_n$ be a finite collection of open subsets of S. Put

$$E = \bigcap_{j=1}^{n} U_j.$$

We wish to show E is open. So let $x \in E$. Then because each U_j is open we can find some $r_j > 0$ so that $B_{r_j}(x) \subseteq U_j$. Now take $r = \min\{r_1, \dots, r_n\}$. Then since $r \le r_j$ for each j we have $B_r(x) \subseteq B_{r_j}(x) \subseteq U_j$ for each j. This shows $B_r(x) \subseteq E$, so x is interior to E, meaning E is open.

13.5 (a) Verify one of De Morgan's Laws for sets:

$$\bigcap\{S \setminus U \,:\, U \in \mathcal{U}\} = S \setminus \bigcup\{U \,:\, U \in \mathcal{U}\}.$$

Solution: Let $x \in \bigcap\{S \setminus U \,:\, U \in \mathcal{U}\}$. Then we have $x \in S \setminus U$ for every $U \in \{U\}$. But this means $x \notin U$ for all $U \in \mathcal{U}$. So $x \notin \bigcup\{U \,:\, U \in \mathcal{U}\}$, which means $x \in S \setminus \bigcup\{U \,:\, U \in \mathcal{U}\}$.

On the other hand, let $x \in S \setminus \bigcup\{U \,:\, U \in \mathcal{U}\}$. Then $x \notin \bigcup\{U \,:\, U \in \mathcal{U}\}$, meaning $x \notin U$ for all $U \in \mathcal{U}$. But this means $x \in S \setminus U$ for every $U \in \mathcal{U}$, which in turn means $x \in \bigcap\{S \setminus U \,:\, U \in \mathcal{U}\}$.

(b) Show that the intersection of any collection of closed sets is a closed set.

Solution: Let $\mathcal{E}$ be an arbitrary family of closed sets in S. Then by definition we have $S \setminus E$ is open for every $E \in \mathcal{E}$. So take $\mathcal{U} = \{S \setminus E \,:\, E \in \mathcal{E}\}$, which is a family of open sets, and note the reciprocal relationship $\mathcal{E} = \{S \setminus U \,:\, U \in \mathcal{U}\}$, since for each $E \in \mathcal{E}$ we have $E = S \setminus (S \setminus E)$. Then we have

$$\begin{aligned}\bigcap\{E \in \mathcal{E}\} &= \bigcap\{S \setminus U \,:\, U \in \mathcal{U}\} \\ &= S \setminus \bigcup\{U \,:\, U \in \mathcal{U}\},\end{aligned}$$

where the second line follows by De Morgan's law. Taking complements on both sides, this shows

$$S \setminus \bigcap\{E \in \mathcal{E}\} = \bigcup\{U \,:\, U \in \mathcal{U}\}.$$

Since we know arbitrary unions of open sets are open, we see that the right hand side above is open, meaning the complement of the intersection of sets in $\mathcal{E}$ is open, which by definition means the intersection itself is closed.

13.6 Prove Proposition 13.9 [page 88 in the textbook].

Solution: For part (a), we wish to show E is closed if and only if $E = E^-$. So suppose E is closed. We clearly have $E \subseteq E^-$ in general since E^- is defined as an intersection of sets all containing E. So we just need to show $E^- \subseteq E$. So let $x \in E^-$. Then x is in every closed set containing E. But since E itself is a closed set, then this means we must have $x \in E$. So indeed $E^- = E$.

To prove the converse, we will now assume $E = E^-$. Then it follows that E is closed because E can be written as the intersection of closed sets (since $E = E^-$) and we know from part (b) of 13.5 above that any such intersection is closed.

For part (b), we will first assume E is closed and show it contains the limit of any convergent sequence in E. So let (x_n) be a convergent sequence of points in E, say $x_n \to x$. Suppose for the sake of contradiction $x \notin E$, meaning $x \in S \setminus E$. Since E is closed, we must have that $S \setminus E$ is open. So then we can find $r > 0$ so that $B_r(x) \subseteq S \setminus E$. But then this means that $x_n \notin B_r(x)$ for any n, i.e. $d(x_n, x) \geq r$ for all n, which contradicts the assumption that $x_n \to x$. So in fact we must have had $x \in E$.

For the converse, suppose E contains the limits of all convergent sequences in E. Then we will show that $S \setminus E$ is open. So let $x \in S \setminus E$. Now suppose it were the case that for every $r > 0$ the intersection $B_r(x) \cap E$ was nonempty. Then we could define a sequence as follows: for each n, take $x_n \in B_{1/n}(x) \cap E$. Then (x_n) would form a sequence in E, and we would have $x_n \to x$ because $d(x_n, x) < 1/n$ for each n. But then by our assumption that E contains the limit of any such sequence, we would need $x \in E$, contradicting the assumption that $x \in S \setminus E$. This means there is some $r > 0$ for which $B_r(x) \cap E = \emptyset$. But this is the same as saying $B_r(x) \subseteq S \setminus E$ for that value of r. This shows that every $x \in S \setminus E$ is interior, meaning $S \setminus E$ is open, so that E is closed.

For part (c), we will first assume $x \in E^-$ and show it is the limit of a sequence of points in E. Note that it would be sufficient to prove the following claim: for any $r > 0$, we have that $B_r(x) \cap E$ is not empty. If our claim is true, then (as in part (b)) we can take $r = 1/n$ and find a point $x_n \in B_r(x) \cap E$. By doing this for each n, we will construct a sequence (x_n) in E which must approach x because we would have $d(x, x_n) < 1/n$ for each n. So let $r > 0$. Now suppose for the sake of contradiction that it were in fact the case that $B_r(x) \cap E$ were empty. Then this would mean that $E \subseteq S \setminus B_r(x)$. But now notice that $S \setminus B_r(x)$ is closed because $B_r(x)$ is open. So then $S \setminus B_r(x)$ is a closed set containing E, meaning we must have $E^- \subseteq S \setminus B_r(x)$. But since $x \in E^-$ we have $x \in S \setminus B_r(x)$ and $x \in B_r(x)$ at the same time, a contradiction. So indeed $B_r(x) \cap E$ is nonempty for all $r > 0$, meaning we can construct a sequence as described earlier.

For the converse, suppose x is the limit of a sequence (x_n) of points in E. Then since we know from part (a) that E^- is closed, by part (b) we know that E^- must contain the limit of any sequence of points in E^-. Clearly since $x_n \in E$ for all n we also have $x_n \in E^-$ for all n so (x_n) is a sequence of points in E^-. So its limit x must be in E^- as well.

For part (d), we will first assume x is in the boundary of E, meaning x is in E^- and x is *not* interior to E. The fact that $x \in E^-$ is literally that x belongs to the closure of E, so we need only show that x also belongs to the closure of $S \setminus E$. For this, we will argue that x belongs to any closed set containing $S \setminus E$. So let $F \supseteq S \setminus E$ be closed. Then note that $S \setminus F$ is open and $S \setminus F \subseteq E$. Now if we had $x \in S \setminus F$ then because $S \setminus F$ is open we could find $r > 0$ so that $B_r(x) \subset S \setminus F$. But because $S \setminus F \subseteq E$ this would mean $B_r(x) \subseteq E$, which would mean x is interior to E, contradicting the assumption that x is a boundary point of E. So in

$x \notin S \setminus F$, meaning $x \in F$. Since F was an arbitrary closed set containing $S \setminus E$, we see that x is in every closed set containing $S \setminus E$, which means $x \in (S \setminus E)^-$, as desired.

For the converse, suppose $x \in E^-$ and $x \in (S \setminus E)^-$. To show x is in the boundary of E, we need only show that x is not interior to E, since we already know $x \in E^-$. Suppose for the sake of contradiction that x were interior to E. Then we could find $r > 0$ so that $B_r(x) \subseteq E$. But then $S \setminus B_r(x)$ would be a closed set with $S \setminus B_r(x) \supseteq S \setminus E$, so we would need $x \in S \setminus B_r(x)$ because we assumed $x \in (S \setminus E)^-$. This is clearly impossible since $x \in B_r(x)$. So indeed x cannot be interior to E, meaning x is in the boundary of E.

13.10 Show that the interior of each of the following sets is the empty set.

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Solution: To show that a point $x \in E$ is not an interior point of E, we must argue that for *every* radius $r > 0$ we have that the ball $B_r(x) = \{y \in S : d(x, y) < r\}$ is not a subset of E. So take $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $x \in E$, say $x = 1/n$ for some $n > 1$. Now let $r > 0$. Then take

$$y = x + \min\left\{\frac{r}{2}, \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right)\right\}.$$

If $n = 1$ then just take $y = 1 + r/2$. Then we see that $x < y < x + r$ meaning $y \in B_r(x)$. Also, since $1/n < y < 1/(n-1)$, we see that $y \notin E$ (or in the case $n = 1$ we have $y > 1$ so $y \notin E$), meaning $B_r(x)$ is not a subset of E. Since r was arbitrary, this shows x is not an interior point. Since x was arbitrary in E, we see that the interior of E is empty.

(b) $\mathbb{Q}$, the set of rational numbers.

Solution: Let $x \in \mathbb{Q}$ and let $r > 0$. Then by denseness of the irrational numbers, we can find an irrational $y \in (x, x + r)$, meaning $y \in B_r(x)$ but $y \notin \mathbb{Q}$, showing $B_r(x)$ is not a subset of $\mathbb{Q}$, so since r was arbitrary we have x is not an interior point of $\mathbb{Q}$, and since x was arbitrary in $\mathbb{Q}$ we see the interior of $\mathbb{Q}$ is empty.

(c) The Cantor set in Example 5 [page 90 in the textbook].

Solution: Let E be the Cantor set and $x \in E$. Now let $r > 0$. Choose n so that $(1/3)^n < r$. Let $F_0 \supset F_1 \supseteq F_2 \supseteq \cdots$ be the chain of closed sets in the standard construction of the Cantor set (i.e. $F_0 = [0, 1]$, $F_1 = [0, 1/3] \cup [2/3, 1]$, etc.). Then F_n is a union of disjoint closed intervals of length $(1/3)^n$. Since $x \in F_n$, let $[a, a + (1/3)^n]$ be the unique closed interval in F_n containing x. Then since $r > (1/3)^n$, clearly $B_r(x)$ is not contained in $[a, a + (1/3)^n]$, since we can for instance find a point in the interval $(a + (1/3)^n, x + r)$. So this means $B_r(x)$ is not a subset of the intersection of all the F_j, which is the Cantor set E, and we can make this argument for any $r > 0$. So no point $x \in E$ is interior to E.

13.12 Let (S, d) be any metric space. [Max's note: the fact that S is a metric space is not exactly important here. You can make simple arguments for both parts of this problem which do not make any reference to the metric d whatsoever, and therefore hold for an arbitrary *topological* space S.]

(a) Show that if E is a closed subset of a compact set F, then E is also compact.

Solution: Let $F \subseteq S$ be compact and $E \subseteq F$ be closed. Let $\mathcal{U}$ be an open cover of E. Now we would like to exploit the fact that F is compact somehow, but to do that we need some open cover of F. Of course, for this to be useful we would need to cover F with something that resembles $\mathcal{U}$ as closely as possible. The issue is that while $\mathcal{U}$

covers E, it may not cover F. But this can be fixed easily with the addition of a single open set, namely $S \setminus E$. Let $\mathcal{U}' = \mathcal{U} \cup \{S \setminus E\}$. Then in fact $\mathcal{U}'$ covers S so it certainly covers F. This means we can find a finite subcover $\mathcal{U}'' \subset \mathcal{U}'$ which covers F, because F is compact. Now there are two cases: either $\mathcal{U}''$ includes $S \setminus E$ as an element or not. In the latter case, then clearly $\mathcal{U}'' \subseteq \mathcal{U}$ is our desired finite subcover of E. In the former case, we argue that we can simply omit $S \setminus E$ from $\mathcal{U}''$ to obtain a finite subcover. To see this, say $\mathcal{U}'' = \{S \setminus E, U_1, U_2, \dots U_n\}$ where each $U_j \in \mathcal{U}$. Then we are claiming that

$$E \subseteq \bigcup_{j=1}^{n} U_j.$$

To see this, let $x \in E$. Then $x \in F$. Since $\mathcal{U}''$ covers F, we must have that x is an element of one of the open sets in $\mathcal{U}''$. But x cannot be in $S \setminus E$ by assumption that $x \in E$, so we must in fact have $x \in U_j$ for some j. This shows that $\{U_1, \dots, U_n\}$ is the desired finite subcover. So in any case we see that an arbitrary open cover of E admits a finite subcover, meaning E is compact.

(b) Show that the finite union of compact sets in S is compact.

Solution: Let $E_1, \dots, E_n$ be a finite collection of compact subsets of S. Let $\mathcal{U}$ be an open cover of $\bigcup_j E_j$. Then since each E_j is compact, we can take a finite subcover $\mathcal{U}_j \subset \mathcal{U}$ for each. Then clearly collecting all the sets in these finitely many covers, i.e. defining

$$\mathcal{U}' = \bigcup_{j=1}^{n} \mathcal{U}_j$$

gives a finite subfamily of $\mathcal{U}$ which covers $\bigcup_j E_j$, and since $\mathcal{U}$ was arbitrary we have shown that $\bigcup_j E_j$ is compact.

14.3 Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{1}{\sqrt{n!}}$

Solution: We use the ratio test and see that $\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{\frac{1}{n+1}} \to 0$, meaning the series converges.

(b) $\sum \frac{2+\cos n}{3^n}$

Solution: Note that the terms of the series are all nonnegative, so we can use the comparison test and argue that

$$\frac{2+\cos n}{3^n} \leq \frac{3}{3^n} = \frac{1}{3^{n-1}},$$

and clearly $\sum 3^{-(n-1)}$ is a convergent geometric series, so the original series converges by comparison.

(c) $\sum \frac{1}{2^n+n}$

Solution: Here we again have all nonnegative terms. We expect the series to converge because the terms asymptotically decay exponentially. In fact, we can make the simple comparison

$$\frac{1}{2^n+n} \leq \frac{1}{2^n},$$

and clearly $\sum 2^{-n}$ is a convergent geometric series, so the original series converges as well.

(d) $\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right)$
Solution: Asymptotically, the factor $(50+2/n)$ approaches 50 so we expect it shouldn't affect the convergence in the presence of the factor of $(1/2)^n$. We can make the comparison

$$\left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right) \leq \left(\frac{1}{2}\right)^n \cdot 52 = \frac{52}{2^n}.$$

Clearly the series $\sum 52/2^n$ is a convergent geometric series, so the original series converges as well.

(e) $\sum \sin\left(\frac{n\pi}{9}\right)$
Solution: We see that whenever n is of the form $n = 3+18k$ we have $\sin(n\pi/9) = \sqrt{3}/2$, so the sequence of terms cannot converge to zero, meaning the series diverges.

(f) $\sum \frac{(100)^n}{n!}$
Solution: Applying the ratio test, we see

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100}{n+1} \to 0,$$

so the series converges.

14.4 Repeat the above exercise for the following.

(a) $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$
Solution: Since n grows without bound, our instinct is that $(-1)^n$ becomes insignificant in comparison, so that the asymptotic behavior of the terms is similar to $1/n^2$. The latter form a convergent series, so we expect the given series to converge as well. To bound the given series above, we can say

$$\frac{1}{[n+(-1)^n]^2} \leq \frac{1}{(n-1)^2},$$

so the given series is bounded above by $\sum_{n=2}^{\infty} 1/(n-1)^2$. The latter series is in fact exactly equal to $\sum_{n=1}^{\infty} 1/n^2$ by shifting the indices, and we know the latter converges, so the original series converges by comparison.

(b) $\sum[\sqrt{n+1} - \sqrt{n}]$
Solution: In this case, we are fortunate to be able to write a formula for the partial sums. Note that we have

$$s_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \ldots + (\sqrt{n+1} - \sqrt{n}),$$

and after cancelling all intermediate terms we find $s_n = \sqrt{n+1} - 1$. Clearly this means $s_n \to \infty$, so the series does not converge.

(c) $\sum \frac{n!}{n^n}$
Solution: We apply the root test. We see that

$$\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{n}(n!)^{1/n},$$

and we showed in exercise 12.14(b) that the latter sequence converges to $1/e$. Since $1/e < 1$, the root test tells us the series converges.

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Solution: If $\sum a_n$ converges, we must have $a_n \to 0$. This means we can find some N for which $a_n < 1$ for all $n > N$. Then we see that for $n > N$ we have $a_n^p < a_n$ since $a_n < 1$ and $p > 1$. This means we have that the series $\sum_{n=N+1}^{\infty} a_n^p$ converges by comparison to the convergent series $\sum_{n=N+1}^{\infty} a_n$. But then by adding the finite sum $\sum_{n=1}^{N} a_n^p$ we see that $\sum a_n^p$ converges, as desired.

14.12 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Solution: Assume as in the problem that $\liminf |a_n| = 0$. Then we know that for any $\varepsilon > 0$ the set $\{n : |a_n| < \varepsilon\}$ is infinite. So define a subsequence a_{n_k} as follows: for each k, taking $\varepsilon = 1/2^k$ we see that we must be able to find some index n_k for which $|a_{n_k}| < 1/2^k$. By making a choice of such an index for each k, we find a subsequence (a_{n_k}) which satisfies $|a_{n_k}| < 1/2^k$ for each k. Then clearly this sequence is absolutely convergent because we have

$$\sum_k |a_{n_k}| \le \sum_k \frac{1}{2^k} = 1.$$

Since $\sum_k a_{n_k}$ is absolutely convergent, it is convergent as well, meaning our choice of (a_{n_k}) works as intended.

14.13 (c) Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint*: Note $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

Solution: Using the hint, we see we can write

$$\begin{aligned} s_n &= \frac{0}{2^2} + \frac{1}{2^3} + \ldots + \frac{n-1}{2^{n+1}} \\ &= \left(\frac{1}{2^1} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \ldots + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right) \\ &= \frac{1}{2^1} - \frac{n+1}{2^{n+1}}. \end{aligned}$$

Since clearly $(n+1)/2^{n+1} \to 0$, we see that indeed $\lim s_n = 1/2$.

(d) Use (c) to calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: First note that by re-indexing the given series (specifically shifting all the indices by 1) we can write

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=2}^{\infty} \frac{n-1}{2^{n-1}}.$$

Since $(n-1)/2^{n-1} = 0$ when $n = 1$ this also means

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}}.$$

But then note

$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8},$$

so we have that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{8}.$$