Math 104, Summer 2019

PSET #4 (due Tuesday, 7/13/2019)

Problem 13.4. Prove (iii) and (iv) in Discussion 13.7 [page 87 in the textbook].

(i) S is open in S (trivial).

(ii) The empty set $\{\}$ is open in S (trivial).

(iii) The union of **any** collection of open sets is open.

(iv) The intersection of **finitely many** open sets is again an open set.

Solution. **(iii)** As given in Rudin, this is to show that for any collection $\{G_\alpha\}$ of open sets, the union $\bigcup_\alpha G_\alpha$ is open.
To see this, let $G := \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α. Because x is an interior point of G_α, x is also an interior point of G; hence G is open, as desired.

(iv) Now we want to show that for any finite collection $G_1, \ldots, G_n$ of open sets, the finite intersection $\bigcap_{i=1}^n$ is again open. Let $H := \bigcap_{i=1}^n G_i$ be our intersection of finitely many open sets. For any $x \in H$, there exists neighborhoods N_i of x with radii $r_i > 0$ such that $N_i \subset G_i$, where $i = 1:n$. Let us define $r := \min\{r_1, \ldots, r_n\}$, and let N be the neighborhood of x with that radius r. Then $N \subset G_i$ for all $i = 1:n$, ensuring that $N \subset H$, hence H is open.

□

Problem 13.5.

1. Verify one of DeMorgan's Laws for sets:

$$\bigcap \{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup \{U : U \in \mathcal{U}\}.$$

2. Show that the intersection of any collection of closed sets is a closed set.

Solution. **(1)** We'll prove this following equivalent statement (with cleaner notation) as given in Rudin. Let $\{E_\alpha\}$ be a (possibly infinite) collection of sets E_α. Then

$$\bigcap_\alpha (E_\alpha^c) = \left(\bigcup_\alpha E_\alpha\right)^c. \tag{1}$$

Let $A := \bigcap_\alpha (E_\alpha^c)$ and $B := (\bigcup_\alpha E_\alpha)^c$. If $x \in A$, then $x \in E_\alpha^c$ for every α; hence $x \notin E_\alpha$ for any α; hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in (\bigcup_\alpha E_\alpha)^c$. Hence we have $A \subset B$.
If $x \in B$, then $x \notin \bigcup_\alpha E_\alpha$; hence $x \notin E_\alpha$ for any α, and hence $x \in E_\alpha^c$ for every α, so that $x \in \bigcap E_\alpha^c$. Hence we have $A \supset B$, as required to conclude equality in (1) above.
(2) We have from the previous problem that for any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open. We wish to show that the intersection of a (possibly infinite) collection $\{F_\alpha\}$ of closed sets is itself closed. That is, $\bigcap_\alpha F_\alpha$ is closed. From part (1) above, we have that

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha^c),$$

and F_α^c is thus open, by problem 13.4 above. Because the complement of a complement of a set X is equivalent to X itself, we conclude that $\bigcap_\alpha F_\alpha$ is closed, as desired.

□

Problem 13.6. Prove Proposition 13.9 [page 88 in the textbook]. Let E be a subset of a metric space (S, d). Prove the following:
(a) The set E is closed if and only if $E = E^-$.
(b) The set E is closed if and only if it contains the limit of every convergent sequence of points in E.
(c) An element is in E^- if and only if it is the limit of some sequence of points in E.
(d) A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.

Solution. **(a)** The closure E^- of E is closed. To see this, consider if $p \in S$ and $p \notin E^-$, then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of E^- is therefor open, and hence E^- is closed.
($\Longleftarrow$) Using this, if $E = E^-$, then E is closed as we have shown E^- is closed.
($\Longrightarrow$) Conversely, if E is closed, then the set E' of all limit points of E in S satisfies $E' \subset E$. However, we defined $E^- := E \cup E'$. Hence we have $E^- = E$.

(b) We first prove the ($\Longrightarrow$) forward direction. Suppose E is closed. For contradiction, suppose there exists some sequence (s_n) in E with $s_n \to s \notin E$. Hence $s \in E^c$. Because E is closed, E^c is open and hence $s \in E^c$ is an interior point. Then there is an open ball $N_r(s) \subset E^c$. Due to convergence, we have some N for which $n > N$ implies $d(s_n, s) < r$. Consider that when $n := \lfloor N \rfloor + 2$, we have $d(s_n, s) < r$ which then gives us $s_n \in N_r(s)$ and $s_n \in E^c$. This is a direct contradiction to our construction of sequence (s_n) of elements of E. Hence our supposition must be false, as required for this direction.
Conversely ($\Longleftarrow$), suppose that every subsequent limit of points in E is contained in E. That is, suppose that for every sequence (s_n) in E with $s_n \to s$, we have $s \in E$. Fix some $n \in \mathbb{N}^+$ and define $r := \frac{1}{n}$. Let $x \in E^-$. This ensures existence of a sequence (x_n) where all $x_n \in N_r(x)$ for all n, so $x_n \to x$. Hence we have $x \in E$ for all x and conclude $E^- \subset E$, and thus invoking part (a) above, E is closed.

(c) ($\Longrightarrow$) Let $x \in E^-$. For some $n \in \mathbb{N}$, let $r := \frac{1}{n}$. Then by (b) above, we have some $s \in E$ where $s \in N_r(x)$. Because we can take any n, we have a sequence (x_n) where all $x_n \in N_r(x)$, which asserts that $d(x_n, x) < \frac{1}{n}$ for all n. Then this sequence converges to some limit in E.
($\Longleftarrow$) Conversely, fix one sequence $(x_n) \to x$. Because $E \subset E^-$, our sequence (x_n) is also in E^-, which is closed, so it contains the limit of every subsequent limit as in (b) above, and hence $x \in E^-$.

(d) ($\Longrightarrow$) Let x be in the boundary of E. Then $x \in E^-$ because x is not an interior point; that is, $x \notin E^o$. Hence by (c) above, we have both $x \in (E^c)^-$ and $x \in (E^o)^c$.
($\Longleftarrow$) Now suppose $x \in (E^c)^- = (E^o)^c$ and $x \in E^-$. Hence $x \notin E^o$, so x is a boundary point as it is in the enclosure E^- but $x \notin E^o$, as required.

□

Problem 13.10. Show that the interior of each of the following sets is the empty set.

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}; \qquad \mathbb{Q}; \qquad \text{The Cantor Set.}$$

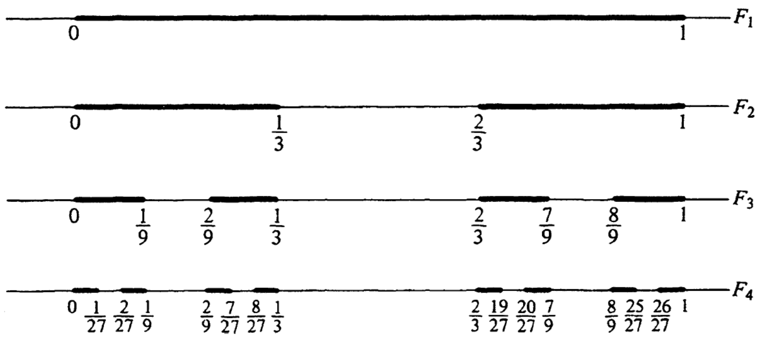

Figure 1: Sketch of the Cantor set. Given by Ross, the Cantor set is a famous nonempty closed set in $\mathbb{R}$ where $F = \bigcap_{n=1}^{\infty} F_n$ where F_n are sketched above. Remarkable properties of this include that the sum of lengths of intervals comprising F_n is equal to $\left(\frac{2}{3}\right)^{n-1}$ and this tends to 0 as $n \to \infty$. Yet the intersection F is so large that it cannot be written as a sequence; that is, it is uncountable. Moreover, the interior of F is the empty set, and thus F is equal to its boundary.

To see the interior of the Cantor set is the empty set, consider associating to each sequence $a = \{\alpha_n\}$, where α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

We can see that the set of all $x(a)$ is precisely the Cantor set. Or more explicitly, let E_0 be the interval $[0, 1]$. Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the intervals $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Continuing as shown in the diagram, we get a sequence of compact sets E_n such that

$$E_1 \supset E_2 \supset E_3 \supset \cdots,$$

where E_n is the union of 2^n intervals, each of length 3^{-n}.
Our set

$$P := \bigcap_{n=1}^{\infty} E_n$$

we define as the Cantor set, and clearly it is compact, with the theorem that:
'If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.'
gives us that P itself is not empty. To show that P has no interior point, it suffices to show P is perfect; that is, to show that P has no isolated point. Let $x \in P$, and let S be any segment containing x. Let I_n be the interval of E_n that contains x. Choose sufficiently large n so that $I_n \subset S$. Let x_n be an endpoint of I_n, such that $x_n \neq x$. It follows from the construction of P that $x_n \in P$; hence we conclude x is a limit point of P and P is perfect. Because x is an arbitrary point in P, x is not interior to the Cantor set; hence we conclude the interior of the Cantor set is the empty set.

We can extend this logic to easier sets like $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $\mathbb{Q}$.

Problem 13.12. Let (X, d) be any metric space. [Max's note: the fact that X is a metric space is not exactly important here. You can make simple arguments for both parts of this problem which do not make any reference to the metric d whatsoever, and therefore hold for an arbitrary *topological* space X.]

1. Show that if F is a closed subset of a compact set K, then F is also compact.
2. Show that the finite union of compact sets in X is compact.

Solution. (1) As given by Rudin, suppose $F \subset K \subset X$, and F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F, and hence we have that F is compact.

(2) To show that the finite union of compact sets in X is compact, we show that if $K \subset Y \subset X$, then K is compact relative to X if and only if K is relative to Y. It easily follows that finite unions of such X, Y are compact relative to X.

Suppose K is compact relative to X and let $\{V_\alpha\}$ be a collection of sets, open relative to Y, such that $K \subset \bigcup_\alpha V_\alpha$. We have proved before that, there are sets G_α ope relative to X such that $V_\alpha = Y \cap G_\alpha$, for all α; and since K is compact relative to X, we have

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

for some choice of finitely many indices $\alpha_1, \ldots, \alpha_n$. Since $K \subset Y$, the above implies

$$K \subset V_{\alpha_1} \cup \because V_{\alpha_n}.$$

This proves K is compact relative to Y.

Conversely, suppose K is compact relative to Y, let $\{G_\alpha\}$ be a collection of open subsets of X that covers K, and set $V_\alpha := Y \cap G_\alpha$. Then for some choice $\alpha_1, \ldots, \alpha_n$ will give us $K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$ and because $V_\alpha \subset G_\alpha$, this implies $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$, as required.

$\square$

Problem 14.3, 14.4 Determine which of the following series converge. Justify your answers.

$$\sum \frac{1}{\sqrt{n!}}; \qquad \sum \frac{2+\cos n}{3^n}; \qquad \sum \frac{1}{2^n+n}; \qquad \sum \left(\frac{1}{2}\right)^n \left(50+\frac{2}{n}\right); \qquad \sum \sin\left(\frac{n\pi}{9}\right); \qquad \sum \frac{(100)^n}{n!};$$

$$\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}; \qquad \sum [\sqrt{n+1}-\sqrt{n}]; \qquad \sum \frac{n!}{n^n}$$

Because this will (probably not) be one of the two graded problems, let's 'prove' convergence using a calculator :-)

```
## Create sum function:

```{r}
sum <- function(a,j) {
  s = 0
  for (i in 1:j) {
    s <- s + a(i)
  }
  s
}
```

Defining sequences
```{r}

a1 <- function(n) 1/sqrt(factorial(n))
a2 <- function(n) (2 + cos(n))/(3^n)
a3 <- function(n) 1/(2^n + n)
a4 <- function(n) (0.5^n) * (50 + 2/n)
a5 <- function(n) sin(n*pi/9)
a6 <- function(n) 100^n / factorial(n)

a7 <- function(n) (n + (-1)^n)^(-2)
a8 <- function(n) (sqrt(n+1) - sqrt(n))
a9 <- function(n) (factorial(n)/n^n)

```

```

Naively, this gives:

$$\sum \frac{1}{\sqrt{n!}} \approx 2.469506; \qquad \sum \frac{2+\cos n}{3^n} \approx 1.091875 \qquad \sum \frac{1}{2^n+n} \approx 0.6972766$$

$$\sum \left(\frac{1}{2}\right)^n \left(50+\frac{2}{n}\right) \approx 51.38629 \qquad \sum \sin\left(\frac{n\pi}{9}\right) \approx \text{N/A (oscillating; neither converge nor diverge)}$$

$$\sum \frac{(100)^n}{n!} \approx \text{N/A; might converge}$$

$$\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2} \approx 0.6449321$$

$$\sum [\sqrt{n+1}-\sqrt{n}] \approx \text{does not seem to converge even at 90million terms}$$

$$\sum \frac{n!}{n^n} \approx \text{our 'calculator test' gives no information :-(}$$

If our above efforts are insufficient, consider the following on the next page:

**Solution.** (1) $\sum \frac{1}{\sqrt{n!}}$ converges. First we notice that for all $n \geq 6$, $\sqrt{n!} > \sqrt{n^3} = n$. We can formalize with induction easily but leave the proof to the reader. Hence for $n \geq 6$,

$$\frac{1}{\sqrt{n!}} < \frac{1}{n}.$$

Further notice that all terms in our summation are strictly positive. Hence we have:

$$\begin{aligned}\sum \frac{1}{\sqrt{n!}} &= \sum \left[\frac{1}{1!} + \frac{1}{\sqrt{2!}} + \frac{1}{\sqrt{3!}} + \frac{1}{\sqrt{4!}} + \frac{1}{\sqrt{5!}}\right] + \sum_{n=6}^{\infty} \frac{1}{\sqrt{n!}} \\ &< \left[\frac{1}{1!} + \frac{1}{\sqrt{2!}} + \frac{1}{\sqrt{3!}} + \frac{1}{\sqrt{4!}} + \frac{1}{\sqrt{5!}}\right] + \sum_{n=6}^{\infty} \frac{1}{\sqrt{n^3}},\end{aligned}$$

and by a comparison test (p-series? t-series? pewdiepie?), we have $p = 1.5 > 1$ and we have convergence as desired. □

**Solution.** (2) The series $\sum \frac{2+\cos n}{3^n}$ converges. To see this, simply consider $1 < 2 + \cos n < 3$, where strict inequality follows from $\pi$ irrational and hence is not an integral multiple of natural numbers (i.e. for no $k \in \mathbb{N}$ do we have $\cos k = \pm 1$). Consider:

$$\sum \frac{1}{3^n} < \sum \frac{2 + \cos n}{3^n} < 3 \cdot \sum \frac{1}{3^n}.$$

We know the quantities at the ends to converge, so by the comparison test (and particularly the second inequality), our series converges. And of course, all terms in our series are strictly positive.

□

**Solution.** (3) $\sum \frac{1}{2^n+n}$ converges. Consider:

$$0 < \sum \frac{1}{2^n + n} < \sum \frac{1}{2^n},$$

and we know the series on the right to converge, and all terms of the series are positive, so by term-wise comparison, we have that our series $\sum \frac{1}{2^n+n}$ converges as desired. □

**Solution.** (4) $\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right)$ converges. Now looking at $\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right)$, we again first notice that all terms in our series are positive. Then we have:

$$\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right) < 52 \sum \left(\frac{1}{2}\right)^n,$$

and we know the expression on the right to converge. Hence our original series converges. □

**Solution.** (5) $\sum \sin\left(\frac{n\pi}{9}\right)$ does not converge. We immediately notice the terms of the summation oscillate and do not themselves converge to 0, so we conclude that our series does not converge.

□

**Solution.** (6) $\sum \frac{(100)^n}{n!}$ converges. As a rare occurrence where the ratio test helps us, define $a_n := \frac{100^n}{n!}$. Consider:

$$\begin{aligned}\limsup \left|\frac{a_{n+1}}{a_n}\right| &= \limsup \left|\frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n}\right| \\ &= \limsup \left|\frac{100}{n+1}\right| = 0 < 1,\end{aligned}$$

and we verify that all terms are nonzero, so we have convergence (albeit slow and very large numbers), as desired. □

**Solution.** (7) $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$ converges. Writing out terms, we can see this is precisely:

$$\sum_{n=2}^{\infty} = \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{4^2} + \cdots ,$$

where spotting the intuitive pattern is valid, as all terms in the series are strictly positive. We conclude

$$\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2} = \sum_{n=2} \frac{1}{n^2},$$

where we know the right summation converges, and hence our original series converges.

□

**Solution.** (8) $\sum[\sqrt{n+1} - \sqrt{n}]$ does not converge. We notice:

$$\begin{aligned} \sum[\sqrt{n+1} - \sqrt{n}] &= \sum \left[\frac{1}{\sqrt{n+1} + \sqrt{n}}\right] \\ &> \sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}}, \end{aligned}$$

where all our terms are strictly positive and by a comparison test ($p$-series, $p = 0.5 < 1$), we know the right expression to diverge; hence our original series diverges. □

**Solution.** (9) $\sum \frac{n!}{n^n}$ does not converge. By inspection, we see that each term of the series is strictly positive. Let us define $a_n := \frac{n!}{n^n}$. By the root test,

$$\limsup |a_n|^{1/n} = \limsup \left(\frac{n!}{n^n}\right)^{1/n} > 1,$$

hence we conclude that our original sequence diverges.

□

**Problem 14.7.** Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

**Solution.** Set $m := \max\{a_n\}$, as $a_n \to 0$. From convergence, we also know that there exists some $N$ for which $n > N$ implies $0 \leq a_n < 1$. Then consider:

$$\sum a_n^p = \sum_{n=1}^{N} a_n^p + \sum_{n=N+1}^{\infty} a_n^p \leq \sum_{n=1}^{N} a_n^p + \sum_{n=N+1}^{\infty} m^p$$

The left term is convergent by construction as it is a finite sum. Then in the right term, all $0 \leq a_n < 1$, so we bound the series term-wise by $m$, and applying the $p$-series (pewdiepie > t-series) test and $p > 1$ by hypothesis, we conclude that $\sum a_n^p$ converges. □

**Problem 14.12.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

**Solution.** Because $\liminf |a_n| = 0$ as given in the hypothesis, we know that there exists a subsequence $(a_{n_j})$ of $(a_n)$ where $\lim_{j\to\infty} |a_{n_j}| = 0$. We devise an algorithm "inductively" to *construct* such a subsequence such that the desired series converges.

**base case** $k = 1$ **:** Because $|a_{n_j}| \to 0$, there exists some $N_1$ where $j > N_1$ implies $|a_{n_j}| < 1$. Let $j_1 > N_1$, and hence $|a_{n_{j_1}}| < 1$.

**recursive definition** $k - 1 \implies k, \forall k > 1$ **:** Because $|a_{n_j}| \to 0$, there exists some $N_k$ where $j > N_k$ implies $|a_{n_j}| < \frac{1}{k^2}$. Let $j_k > \max\{1 + j_{k-1}, N_k\}$. Then $j_k > j_{k-1}$ and $|a_{n_{j_k}}| < \frac{1}{k^2}$.

We can continue this algorithm iteratively (infinitely) to generate a subsequence that satisfies our desired property with $\sum_{k=1}^{\infty} a_{n_k}$ converges, because terms in our summation will be strictly positive but also strictly less than the geometric series (via comparison test).

□

**Problem 14.13.**

(c) Prove that
$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}.$$

(d) Use (c) to calculate
$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

**Solution.** **(c)** Consider:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \sum \frac{2n-n-1}{2^{n+1}} \\ &= \sum \left[\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right] \\ &= \left(\frac{1}{2} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{3}{8}\right) + \left(\frac{3}{8} + \frac{4}{16}\right) + \cdots \\ &= \frac{1}{2} + \left(-\frac{2}{4} + \frac{2}{4}\right) + \left(-\frac{3}{8} + \frac{3}{8}\right) + \cdots \\ &= \frac{1}{2} + 0 = \frac{1}{2},\end{aligned}$$

which was to be shown. Notice we only used pair-wise associativity of addition for neighboring terms and no assumptions of commutativity. Our expression should hold in the infinite series setting. □

**Solution.** **(d)** From above, we have:

$$\begin{aligned}\sum \frac{n}{2^n} &= \sum \frac{n-1}{2^{n+1}} + \sum \frac{n+1}{2^{n+1}} \\ &= \frac{1}{2} + \sum \left[\frac{n-1}{2^{n+1}} + \frac{2}{2^{n+1}}\right] \\ &= \frac{1}{2} + \frac{1}{2} + \sum \left(\frac{1}{2}\right)^n = 2.\end{aligned}$$

□