Math 104: Homework 3

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

10.1 Which of the following sequences are increasing? decreasing? bounded?

(a) $\frac{1}{n}$

Solution: This sequence is (strictly) decreasing, as well as bounded (above by 1 and below by 0).

(b) $\frac{(-1)^n}{n^2}$

Solution: This sequence is neither increasing nor decreasing, but it is bounded (above by $\frac{1}{4}$ and below by -1).

(c) n^5

Solution: This sequence is increasing and unbounded (though bounded below by 0).

(d) $\sin(\frac{n\pi}{7})$

Solution: This sequence is neither increasing nor decreasing, but bounded (above by 1 and below by -1).

(e) $(-2)^n$

Solution: This sequence is neither increasing nor decreasing, and it is unbounded (in both directions).

(f) $\frac{n}{3^n}$

Solution: This sequence is decreasing and bounded (above by $\frac{1}{3}$ and below by 0).

10.6 (a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

Solution: Let $\varepsilon > 0$. Our aim is to find N so that $|s_m - s_n| < \varepsilon$ for any $m, n > N$. To do this, we take advantage of the fact that (assuming WLOG $n \geq m$)

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \ldots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \ldots + |s_{m+1} - s_m| \\
&\leq 2^{-(n-1)} + 2^{-(n-2)} + \ldots + 2^{-(m+1)} + 2^{-m} \\
&\leq 2^{-m} + 2^{-(m+1)} + 2^{-(m+2)} + \ldots \\
&= 2^{-m} \sum_{j=0}^{\infty} 2^{-j} \\
&= 2^{-m} \cdot 2 \\
&= 2^{-(m-1)},
\end{aligned}$$

so if we pick N so that $2^{-(N-1)} < \varepsilon$ then for $m, n > N$ (with $n > m$) we have $2^{-(m-1)} < \varepsilon$, so by the above reasoning we also have $|s_n - s_m| < \varepsilon$.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution: No, the result is not true. For a counterexample, we can take the sequence of partial sums of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n}$, i.e.

$$
\begin{aligned}
s_1 &= \frac{1}{2}, \\
s_2 &= \frac{1}{2}\left(1 + \frac{1}{2}\right), \\
s_3 &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right), \\
s_4 &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right), \\
&\dots
\end{aligned}
$$

Then we see indeed $|s_{n+1} - s_n| = s_{n+1} - s_n = \frac{1}{2(n+1)} < 1/n$, but since the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges then the sequence (s_n) diverges as well, meaning it cannot be Cauchy.

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence of points in S such that $\lim s_n = \sup S$.

Solution: Say $s = \sup S$. Our first task is to construct a sequence s_n that we believe should approach s. Intuitively, we might want to begin by picking $s_1 \in S$ arbitrarily and then repeatedly picking $s_2 > s_1$, $s_3 > s_2$, etc., which we expect should be possible since each $s_j < s$ and therefore cannot be an upper bound on S. However, this approach fails if we "accidentally" do not pick the s_n to "increase enough" – that is, the resulting sequence may approach a limit strictly less than s. So we must find a way to "force" the sequence to approach s. This can be accomplished as follows: Since $s - 1 < s$, we must be able to find s_1 with $s - 1 < s_1 < s$, since $s - 1$ cannot be an upper bound for S. Similarly, since $s - \frac{1}{2} < s$, we must be able to find s_2 with $s - \frac{1}{2} < s_2 < s$ since $s - \frac{1}{2}$ cannot be an upper bound for S. Continuing in this fashion, we see we must be able to find s_n with $s - \frac{1}{n} < s_n < s$ for every $n \in \mathbb{N}$. If we choose s_n in this fashion for each n then we expect the resulting sequence to converge to s. Indeed, we can argue $s_n \to s$ very easily by invoking the squeeze lemma, since $s - \frac{1}{n} < s_n < s$ and $s - \frac{1}{n} \to s$. Note that we technically relied on the assumption $s \notin S$ in asserting that we can choose each $s_n < s$. Of course, if $s \in S$ we could simply take $s_n = s$ (for all n) to furnish a sequence in S converging to s.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

Solution: We have

$$
\begin{aligned}
s_2 &= \frac{1}{3}(s_1 + 1) = \frac{1}{3}(1 + 1) = \frac{2}{3} \\
s_3 &= \frac{1}{3}(s_2 + 1) = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9} \\
s_4 &= \frac{1}{3}(s_3 + 1) = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}
\end{aligned}
$$

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

Solution: We have the base case $s_1 > \frac{1}{2}$ by definition. For the inductive step, suppose $s_k > \frac{1}{2}$ for some particular k. Then we see

$$s_{k+1} = \frac{1}{3}(s_k + 1) > \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{2}$$

as well, completing the proof.

(c) Show (s_n) is a decreasing sequence.

Solution: Let $n \in \mathbb{N}$. Then we have

$$s_{n+1} - s_n = \frac{1}{3}(s_n + 1) - s_n = \frac{1}{3} - \frac{2}{3}s_n < \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{2} = 0,$$

and $s_{n+1} - s_n < 0$ implies $s_{n+1} < s_n$ (for the inequality above we used the result of the previous part where we showed $s_n > \frac{1}{2}$ for all n). Since n was arbitrary, this shows (s_n) is decreasing.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

Solution: By part (b), we have that (s_n) is bounded below by $\frac{1}{2}$. By part (c), since (s_n) is decreasing, we also have that (s_n) is bounded above by $s_1 = 1$. Also by (c), we see that (s_n) is monotonic, so by Theorem 10.2 s_n must converge. We then argue that the limit must satisfy $s = \frac{1}{3}(s+1)$, so that $s = \frac{1}{2}$ as a result.

11.3 Consider the sequences defined as follows:

$$s_n = \cos\left(\frac{n\pi}{3}\right), \quad t_n = \frac{3}{4n+1}, \quad u_n = \left(-\frac{1}{2}\right)^n, \quad v_n = (-1)^n + \frac{1}{n}.$$

(a) For each sequence, give an example of a monotone subsequence.

Solution: For (s_n), we can take the subsequence consisting only of those terms where $n/3$ is a positive even integer, e.g. by taking $n_k = 6k$. Then we see $s_{n_k} = \cos\left(\frac{6k\pi}{3}\right) = \cos(2k\pi) = 1$, so that (s_{n_k}) is constant and therefore monotone (both increasing and decreasing).

For (t_n), we see that the sequence (t_n) itself is already decreasing, so we can simply take (t_n) as a subsequence of itself.

For (u_n), we see that when n is even, the terms are positive integer powers of $\frac{1}{4}$, which form a decreasing subsequence. So taking $n_k = 2k$ we obtain the subsequence of even-numbered terms, that is, $u_{n_k} = \left(-\frac{1}{2}\right)^{2k} = \frac{1}{4^k}$ which is decreasing.

For (v_n), we see that again there is a sort of alternating behavior, so to find a monotone subsequence we might begin by looking at only even- or only odd-numbered terms. Indeed, for n even we have $v_n = 1 + \frac{1}{n}$ which is decreasing, so taking $n_k = 2k$ as we did for (u_n) suffices.

(b) For each sequence, give its set of subsequential limits.

Solution: For (s_n), the set of subsequential limits is simply $\left\{0, \pm\sqrt{\frac{3}{2}}\right\}$ because these are the only three values (s_n) takes, and each is repeated infinitely often.

For (t_n), we see that the sequence actually converges with limit 0, so the set of subsequential limits is the singleton $\{0\}$.

For (u_n), we see that again the sequence converges to 0, so the set of subsequential limits is $\{0\}$.

For (v_n), we see that the subsequence of odd-numbered terms converges to -1 while the subsequence of even-numbered terms converges to 1. This is actually sufficient to conclude that the set of subsequential limits is precisely $\{-1, 1\}$, because any subsequence will either include infinitely many of both the odd- and even-numbered terms (and hence not converge) or will include only finitely many of one of the two subsequences (and hence converge to the same limit as the other).

(c) For each sequence, give its lim sup and lim inf.

Solution: Using the fact that the lim sup and lim inf are the supremum and infimum (respectively) of the sets of subsequential limits, we see that

$$\begin{aligned}
\limsup s_n &= \sup\left\{0, \pm\sqrt{\frac{3}{2}}\right\} = \sqrt{\frac{3}{2}},\\
\liminf s_n &= \inf\left\{0, \pm\sqrt{\frac{3}{2}}\right\} = -\sqrt{\frac{3}{2}},\\
\limsup t_n &= \sup\{0\} = 0,\\
\liminf t_n &= \inf\{0\} = 0,\\
\limsup u_n &= \sup\{0\} = 0,\\
\liminf u_n &= \inf\{0\} = 0,\\
\limsup v_n &= \sup\{-1, 1\} = 1,\\
\liminf v_n &= \inf\{-1, 1\} = -1.
\end{aligned}$$

(d) Which of the sequences converge? diverge to $+\infty$? diverge to $-\infty$?

Solution: We see that (s_n) does not converge, but it does not diverge to $\pm\infty$ since there are finite numbers that occur as subsequential limits. We already noted above that (t_n) and (u_n) converge to 0. The sequence (v_n) does not converge (as it has more than one subsequential limit) but, like (s_n), it has at least one subsequential limit which is finite and therefore does not diverge to $\pm\infty$.

(e) Which of the sequences are bounded?

Solution: All of the sequences are bounded: (s_n) by ± 1, (t_n) by 0 and 1, (u_n) by ± 1 and (v_n) by ± 2.

11.4 Repeat the previous exercise for the sequences

$$w_n = (-2)^n, \quad x_n = 5^{(-1)^n}, \quad y_n = 1 + (-1)^n, \quad z_n = n \cos\left(\frac{n\pi}{4}\right).$$

Solution: For (w_n), we can take the even-numbered terms as an increasing subsequence, i.e. with $n_k = 2k$ we have $w_{n_k} = 4^k$. Since the even-numbered terms diverge to ∞ and the odd-numbered terms diverge to $-\infty$ we have that the set of subsequential limits is $\{\pm\infty\}$. From this we see $\limsup w_n = \infty$ and $\liminf w_n = -\infty$, that the sequence diverges, and that it is neither bounded above nor below.

For (x_n), we can again take $n_k = 2k$, which results in a constant (and hence monotonic) subsequence $x_{n_k} = 5$. Since the even-numbered terms are all 5 and the odd-numbered terms are all $1/5$, we have that the set of subsequential limits is $\{5, 1/5\}$. From this we see that $\limsup x_n = 5$ and $\liminf x_n = 1/5$, meaning the sequence diverges, and it is bounded above and below by 5 and $1/5$ respectively.

For (y_n), we can again take $n_k = 2k$, which results in the constant subsequence $y_{n_k} = 2$. Since the even-numbered terms are all 2 and the odd-numbered terms are all 0, we see that the set of subsequential limits is $\{0, 2\}$, meaning $\limsup y_n = 2$ and $\liminf y_n = 0$, and (y_n) diverges, and it is bounded above and below by 2 and 0, respectively.

For (z_n), we can take $n_k = 8k$, so that $z_{n_k} = 8k\cos\left(\frac{8k\pi}{4}\right) = 8k\cos(2k\pi) = 8k$, which is monotonically increasing. To find the set of subsequential limits, consider the following: the factor $\cos\left(\frac{n\pi}{4}\right)$ takes on exactly five distinct values: $0, \pm\sqrt{1}2$ and ± 1. The terms where the cosine takes on positive values will be of the form $n\sqrt{\frac{1}{2}}$ or simply n, both of which diverge to infinity, while similarly the terms where the cosine takes a negative value will diverge to negative infinity. The remaining terms are 0, which occurs infinitely many times and is therefore also a subsequential limit. This means the set of subsequential limits is $\{0, \pm\infty\}$. From this we see that $\limsup z_n = \infty$, $\liminf z_n = -\infty$ and (z_n) diverges, and (z_n) is neither bounded above nor below.

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).

Solution: Let S denote the set of subsequential limits of (q_n). Clearly for $x < 0$ we cannot have $x \in S$, because we have $q_n > x/2$ for all n so any $s \in S$ must also satisfy $s \geq x/2 > x$. Similarly, any $x > 1$ cannot be in S because for any such x we have $q_n < (1 + x)/2$ for all n so $s \leq (1 + x)/2 < x$ for any $s \in S$. So $S \subseteq [0, 1]$. We claim in fact $S = [0, 1]$. First let $x \in [0, 1]$. Then take $n_1 = 1$ and define n_k inductively as follows: take $n_{k+1} > n_k$ so that $q_{n_{k+1}} \in (0, 1) \cap (x - 1/k, x + 1/k)$. This must be possible because for each k the intersection $(0, 1) \cap (x - 1/k, x + 1/k)$ is an interval of nonzero length and therefore contains infinitely many rational numbers, meaning that the q_n with $n < n_k$ cannot have exhausted all rational numbers in the intersection, so that we must be able to find some $n_{k+1} > n_k$ with $q_{n_{k+1}} \in (0, 1) \cap (x - 1/k, x + 1/k)$. Then the subsequence (q_{n_k}) of (q_n) so defined will converge to x because we have for each k that $|q_{n_k} - x| < 1/k$, so given $\varepsilon > 0$ we can force $|q_{n_k} - x| < \varepsilon$ by taking $k > 1/\varepsilon$. This shows that $[0, 1] \subseteq S$ so indeed we have $S = [0, 1]$.

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

Solution: We have $\limsup q_n = \sup[0, 1] = 1$ and $\liminf q_n = \inf[0, 1] = 0$.

11.9 (a) Show the closed interval $[a, b]$ is a closed set.

Solution: Let (s_n) be a convergent sequence in $[a, b]$, i.e. $s_n \in [a, b]$ for each n. Then by exercise 8.9(c) we have that $\lim s_n \in [a, b]$ as well. This is the criterion for closedness.

(b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

Solution: No, there can be no such sequence, because the set of subsequential limits of any sequence is always closed, but $(0, 1)$ is not a closed set. This is because we can take the sequence $s_n = 1/n$ as an example of a sequence of points in $(0, 1)$ which approaches 0, and the sequence $s_n = 1 - 1/n$ is a sequence in $(0, 1)$ that approaches 1. Since 0 and 1 are not in $(0, 1)$ then $(0, 1)$ is not closed.

12.3 Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots)$$
$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \ldots)$$

Find

(a) $\liminf s_n + \liminf t_n$

Solution: We have $\liminf s_n = 0$ and $\liminf t_n = 0$ so $\liminf s_n + \liminf t_n = 0$ as well.

(b) $\liminf(s_n + t_n)$

Solution: We have $(s_n + t_n) = (2, 2, 2, 1, 2, 2, 2, 1, \ldots)$, so $\liminf(s_n + t_n) = 1$.

(c) $\liminf s_n + \limsup t_n$

Solution: We have $\liminf s_n = 0$ and $\limsup t_n = 2$ so $\liminf s_n + \limsup t_n = 2$.

(d) $\limsup(s_n + t_n)$

Solution: From the description in (b) we see that $\limsup(s_n + t_n) = 2$.

(e) $\limsup s_n + \limsup t_n$

Solution: We have $\limsup s_n + \limsup t_n = 4$.

(f) $\liminf(s_n t_n)$

Solution: We have $(s_n t_n) = (0, 1, 2, 0, 0, 1, 2, 0, \ldots)$ so $\liminf(s_n t_n) = 0$.

(g) $\limsup(s_n t_n)$

Solution: From the description above we see $\limsup(s_n t_n) = 2$.

12.4 Show $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). *Hint:* First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply exercise 9.9(c).

Solution: As in the hint, we will begin by showing $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. To see this, note that for $n > N$ we have $s_n \leq \sup\{s_n : n > N\}$ and similarly $t_n \leq \sup\{t_n : n > N\}$. Adding the two inequalities shows $s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ for all $n > N$, which in turn means that

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\},$$

since we have shown that the right hand side is an upper bound for the set $\{s_n + t_n : n > N\}$. Now note that $\limsup(s_n + t_n)$ is defined as the limit of the left-hand side above as $N \to \infty$, while $\limsup s_n + \limsup t_n$ is the limit of the right-hand side as $N \to \infty$. So by exercise 9.9(c) we have that the limit of the left-hand side should be less than or equal to the limit of the right-hand side, i.e.

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n,$$

as desired.

12.14 Calculate

(a) $\lim(n!)^{1/n}$

Solution: Note that if we define $s_n = n!$, then $\left|\frac{s_{n+1}}{s_n}\right| = n + 1 \to \infty$, so by Corollary 12.3 we see that $\lim |s_n|^{1/n} = \lim(n!)^{1/n} = \infty$ as well.

(b) $\lim \frac{1}{n}(n!)^{1/n}$

Solution: Here we define $s_n = n!/n^n$, giving

$$\left|\frac{s_{n+1}}{s_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n+1}{(n+1)^{n+1}/n^n} = \left(\left(\frac{n+1}{n}\right)^n\right)^{-1} = \left(\left(1 + \frac{1}{n}\right)^n\right)^{-1}.$$

Using the rightmost expression we see that $\left|\frac{s_{n+1}}{s_n}\right| \to e^{-1}$, meaning by Corollary 12.3 that $|s_n|^{1/n} = \frac{1}{n}(n!)^{1/n} \to e^{-1}$ as well.