Math 104, Summer 2019

PSET #3 (due Tuesday 7/9/2019)

[10.1] Which of the following sequences are increasing? decreasing? bounded?

$$a_n := \frac{1}{n} \qquad b_n := \frac{(-1)^n}{n^2} \qquad c_n := n^5 \qquad d_n := \sin\left(\frac{n\pi}{7}\right) \qquad e_n := (-2)^n \qquad f_n := \frac{n}{3^n}$$

Solution. (1) To see a_n is decreasing, we can easily prove via induction that $\forall n \in \mathbb{N}, a_{n+1} < a_n$. Alternatively to see this easily, the sequence $\left(\frac{1}{a_n}\right)$ is known to be monotonic increasing, where $\frac{1}{a_{n+1}} > \frac{1}{a_n}$ for all n. To see it is bounded, consider also that for all $n \in \mathbb{N}^+$,

$$0 < a_n \leq 1,$$

where 0 is a lower bound and 1 is an upper bound for (a_n).

(2) We can easily show $b_n := \frac{(-1)^n}{n^2}$ (is Cauchy and hence) converges and hence is bounded. To see b_n fails to be increasing or decreasing, it suffices to explicitly show:

$$b_1 = -1; \qquad b_2 = \frac{1}{4}; \qquad b_3 = \frac{-1}{9}$$
$$\implies b_1 < b_3 < b_2,$$

and thus b_n is neither increasing nor decreasing.

(3) We can easily show $c_n := n^5$ is strictly (monotonic) increasing, and because of this, it is bounded below by c_1. It is easy to see that supposing an upper bound for c_n leads to contradiction. As c_n is increasing and unbounded, we say c_n diverges to $+\infty$.

(4) Suppose we know $-1 \leq \sin(x) \leq 1$, for all x. Then we conclude $d_n := \sin\left(\frac{n\pi}{7}\right)$ has -1 and -1 as lower and upper bounds, respectively (we make no claims about l.u.b or the such), so we conclude d_n is bounded. As we have done in (2) above, to show our sequence is neither decreasing nor increasing, consider:

$$d_1 = \sin\left(\frac{\pi}{7}\right) > 0; \qquad d_8 = \sin\left(\frac{8\pi}{7}\right) < 0; \qquad d_{14} = \sin\left(\frac{14\pi}{7}\right) = 0$$
$$\implies d_8 < d_{14} < d_1,$$

and hence d_n is neither increasing nor decreasing.

(5) We define $e_n := (-2)^n = (-1)^n (2^n)$. Without citing radius of convergence, it suffices to know for all n, $|e_{n+2}| > |e_n|$, which is sufficient to show it is not bounded. To show it is not monotonic, it suffices to show: $e_1 = -2; \quad e_2 = 4; \quad e_3 = -8$, and hence:

$$e_3 < e_1 < e_2.$$

We conclude e_n is not bounded and not monotonic (neither increasing nor decreasing).

(6) We define $f_n := \frac{n}{3^n}$. As this is less obvious than the above examples, let us formalize our claims here. Consider that for all $n \in \mathbb{N}$, $0 < f_n$, so 0 is a lower bound. Moreover, consider

$$\begin{aligned} \frac{f_{n+1}}{f_n} &= \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \\ &= \frac{n+1}{3n} = \frac{1}{3} + \frac{1}{3n}. \end{aligned}$$

So we have, for all $n \in \mathbb{N}^+$,

$$0 < \frac{1}{3} \le \frac{f_{n+1}}{f_n} \le \frac{2}{3} < 1,$$

and because $f_n > 0$ above, we can multiply the above compound inequality by f_n to get:

$$0 < \frac{1}{3} f_n \le f_{n+1} \le \frac{2}{3} < f_n,$$

and $f_{n+1} < f_n$, as desired to show it is monotonic nonincreasing (decreasing). It directly follows that $s_1 = \frac{1}{3}$ is an upper bound, and we conclude f_n is decreasing and bounded (and hence converges).

$\square$

[10.6]
1. Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.
2. Is the result in (1) above true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution. **(1)** We have that a sequence is Cauchy if and only if it is convergent, so as the problem states, if we show (s_n) is Cauchy, then we have convergence for free.
Let $\epsilon > 0$. To prove (s_n) is Cauchy, we must show that for any such ϵ, there exists some N for which $n, m > N$ implies $|s_n - s_m| < \epsilon$. Notice that if $n = m$, this result is trivial. WLOG, we then assume $n < m$, so that for some $k \in \mathbb{N}^+, n + k = m$. Define $N := \left|\log_{1/2}\left[\frac{\epsilon}{k}\right]\right|$ and choose any such $n, m > N$ and let them be fixed. Then $n, k \in \mathbb{N}$ are fixed. Consider:

$$\begin{aligned}(n > N) \implies \left(n > \left|\log_{1/2}\left[\frac{\epsilon}{k}\right]\right|\right) \implies |s_{n+k} - s_n| &< k\left(\frac{1}{2}\right)^n \quad \text{(by Lemma 0.1 below)} \\ &< k\left(\frac{1}{2}\right)^{\left|\log_{1/2}[\epsilon/k]\right|} \\ &= k\left[\epsilon/k\right] = \epsilon,\end{aligned}$$

precisely as required to show (s_n) is Cauchy, and hence it is convergent. □

Lemma 0.1. To help with our above proof, given $|s_{n+1} - s_n| < 2^{-n}$, we claim:

$$\forall n, k \in \mathbb{N}^+, \qquad |s_{n+k} - s_n| < k\left(\frac{1}{2}\right)^n.$$

Proof. We formalize with induction. Consider two sets: the set $U \subset \mathbb{N}$ of k and the set $V \subset \mathbb{N}$ of n for which our claim is true. First, fix $k = 1$. Then for all n, our claim is simply the given condition $|s_{n+k} - s_n| = |s_{n+1} - s_n| < 2^n$, which is guaranteed for all n. So we conclude that for $k = 1$, $V = \mathbb{N}$ by the induction axiom. Because we have all n for $k = 1$, if we can perform induction on k for fixed n, then we have that our claim is true for all such pairs n, k. That is, for fixed n, we show $1 \in U$ and $k \in U \implies (k+1) \in U$, to be able to conclude $U = \mathbb{N}$.
Fix $n \in \mathbb{N}$. If $k = 1$, this is simply our given condition, so $1 \in U$. Now we assume $k \in U$, so that for fixed n and k, we have

$$|s_{n+k} - s_n| < k\left(\frac{1}{2}\right)^n.$$

Then consider:

$$
\begin{aligned}
|s_{n+(k+1)} - s_n| &= |s_{n+(k+1)} + (s_{n+k} - s_{n+k}) - s_n| \\
&\leq |s_{n+(k+1)} - s_{n+k}| + |s_{n+k} - s_n| \quad \text{(triangle inequality, twice)} \\
&< |s_{n+(k+1)} - s_{n+k}| + k\left(\frac{1}{2}\right)^n \quad \text{(inductive step, } k \in U\text{)} \\
&\leq \left(\frac{1}{2}\right)^{n+k} + k\left(\frac{1}{2}\right)^n \quad (n+k \in V \text{ as given above, or by the given condition)} \\
&= \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^k + k\left(\frac{1}{2}\right)^n \\
&= \left(\frac{1}{2}\right)^n \left[\underbrace{\left(\frac{1}{2}\right)^k}_{<1} + k\right] \\
&< \left(\frac{1}{2}\right)^n [k+1],
\end{aligned}
$$

which is our claim for $k+1$, as desired to show $k \in U \implies (k+1) \in U$. Hence we have shown for $k = 1$, we have all $n \in \mathbb{N}$. And we have shown that for any n, $k = 1$ is guaranteed, and $k \implies (k+1)$. So by induction, we conclude that we have proven our claim for all $n, k \in \mathbb{N}$.

□

Solution. **(2)** We are given $|s_{n+1} - s_n| < \frac{1}{n}$, for all $n \in \mathbb{N}$.
Let us define

$$s_n := \sum_{i=1}^{n} \frac{1}{i}.$$

Surely,

$$|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n},$$

for all $n \in \mathbb{N}$, so our definition for s_n is valid for our given condition. However, it's easy to prove that our defined s_n diverges to $+\infty$ (or we can take it to be commonly known). □

[10.7] Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence s_n of points in S such that $\lim s_n = \sup S$.

Solution. We are given S is bounded and nonempty, which gives $M, N \in \mathbb{R}$ such that for all $s \in S$, $M \leq s \leq N$. Moreover, we suppose $\sup S \notin S$. We proceed in the most direct way.
Because $\sup S$ is by definition the least upper bound of S, $(\sup S - \frac{1}{n})$ cannot be an upper bound for S, so there exist $s \in S$ such that for all $n \in \mathbb{N}$,

$$\sup S - \frac{1}{n} < s < \sup S.$$

The fact that $\sup S \notin S$ gives the existence of such s satisfying the above. Defining some sequence s_n of such s, and applying $\lim_{n\to\infty}$ across the inequality, we have:

$$\begin{aligned} \lim_{n\to\infty}\left[\sup S - \frac{1}{n}\right] = \lim_{n\to\infty} \sup S - \lim_{n\to\infty} \frac{1}{n} &\leq \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} \sup S \\ \implies \sup S &\leq \lim_{n\to\infty} s_n \leq \sup S, \end{aligned}$$

which directly shows $\lim_{n\to\infty} s_n = \sup S$, as required. $\square$

[10.10] Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

1. Find s_2, s_3 and s_4.
2. Use induction to show $s_n > \frac{1}{2}$ for all n.
3. Show (s_n) is a decreasing sequence.
4. Show $\lim s_n$ exists and find $\lim s_n$.

Solution. **(1)** We simply perform the given iteration to yield:

$$\begin{aligned} s_2 &= \frac{1}{3}(s_1+1) = \frac{1}{3}(1+1) = \frac{2}{3} \\ s_3 &= \frac{1}{3}(s_2+1) = \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{3^2} \\ s_4 &= \frac{1}{3}(s_3+1) = \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{14}{3^3}. \end{aligned}$$

(2) To show $s_n > \frac{1}{2}$ for all $n \geq 1$, we formalize using induction. Consider the set $U \subset \mathbb{N}$ of n for which this inequality holds. Surely, $s_1 = 1 > \frac{1}{2}$, so $1 \in U$. Moreover, $2, 3, 4 \in U$ as can be seen explicity above.
Assume $k \in U$, so that $s_k > \frac{1}{2}$. By our recursive definition, we have:

$$\begin{aligned} s_{k+1} &= \frac{1}{3}(s_k+1) = \frac{s_k}{3} + \frac{1}{3} \\ &> \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3} \\ &= \frac{1}{2}, \end{aligned}$$

precisely as need to show $k \in U \implies (k+1) \in U$. Hence by the induction axiom, we conclude $U = \mathbb{N}$ as desired.
(3) To show (s_n) is a decreasing sequence, we want to show that for all $n \in \mathbb{N}$, $s_{n+1} \leq s_n$.
We can proceed with induction on n, but because this is Analysis, suppose $s_{n+1} > s_n$ for some n, for contradiction. By our iterative definition, we have:

$$s_{n+1} := \frac{1}{3}(s_n+1) = \frac{1}{3}s_n + \frac{1}{3}$$

Substituting this expression in for s_{n+1} in our inequality, we then have:

$$\begin{gathered} \frac{1}{3}s_n + \frac{1}{3} > s_n \\ 2s_n < 1 s_n < \frac{1}{2}, \end{gathered}$$

a contradiction to our finding in (b) that for all $n \in \mathbb{N}$, $s_n > \frac{1}{2}$. Thus our supposition must be false that there exists some n for which $s_{n+1} > s_n$, and so we conclude that s_n is monotonic nonincreasing (decreasing).
(4) Now we show $\lim s_n$ exists and find its value. We have shown s_1 is monotonic decreasing, so we have 1 as an upper bound. We also found that $s_n > \frac{1}{2}$ for all n, so we have $\frac{1}{2}$ as a lower bound and conclude s_n is bounded, monotonic decreasing. Hence it is easy to see

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} \left(\inf\{s_k : k \geq n\}\right) = \frac{1}{2}.$$

Additionally, we have:

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sup\{s_k : k \geq n\}\right) = \frac{1}{2}.$$

Hence because

$$\liminf_{n\to\infty} = \frac{1}{2} \leq \lim_{n\to\infty} s_n \leq \frac{1}{2} = \limsup_{n\to\infty},$$

we conclude $\lim s_n = \frac{1}{2}$. □

[11.3, 11.4] Consider the sequences defined as follows, and answer a set of questions.

Solution. We write out the first few terms of each sequence:

$$
\begin{aligned}
s_n &:= \cos\left(\frac{n\pi}{3}\right) = \left\{\frac{1}{2}, \frac{-1}{2}, -1, \frac{-1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \dots\right\} \\
t_n &:= \frac{3}{4n+1} = \left\{\frac{3}{5}, \frac{3}{9}, \frac{3}{13}, \frac{3}{17}, \dots\right\} \\
u_n &:= \left(-\frac{1}{2}\right)^n = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \dots\right\} \\
v_n &:= (-1)^n + \frac{1}{n} = \left\{0, 1+\frac{1}{2}, -1+\frac{1}{3}, 1+\frac{1}{4}, -1+\frac{1}{5}, \dots\right\} \\
w_n &:= (-2)^n = \{-2, 4, -8, 16, -32, 64, \dots\} \\
x_n &:= 5^{(-1)^n} = \left\{\frac{1}{5}, 5, \frac{1}{5}, 5, \dots\right\} \\
y_n &:= 1 + (-1)^n = \{0, 2, 0, 2, \dots\} \\
z_n &:= n\cos\left(\frac{n\pi}{4}\right) = \left\{1\left(\frac{\sqrt{2}}{2}\right), 2\cdot 0, 3\left(\frac{-\sqrt{2}}{2}\right), 4\left(-1\right), 5\left(\frac{-\sqrt{2}}{2}\right), 6\cdot 0, 7\left(\frac{\sqrt{2}}{2}\right), 8\cdot 1, 9\left(\frac{\sqrt{2}}{2}\right), \dots\right\}
\end{aligned}
$$

For each sequence, give an example of a monotonic subsequence.

$$
\begin{aligned}
(s_{n_k}) :=& \{s_6, s_{12}, s_{18}, \dots, s_{6k}, \dots\} \\
=& \left\{\cos\left(\frac{6\pi}{3}\right), \cos\left(\frac{12\pi}{3}\right), \cos\left(\frac{18\pi}{3}\right), \dots\right\} \\
=& \{1, 1, 1, \dots\} \qquad \text{(constant, so monotonically increasing and also monotonically decreasing)} \\
(t_{n_k}) :=& \{t_1, t_2, t_3, \dots, t_k, \dots\} = t_n \\
=& \left\{\frac{3}{5}, \frac{3}{9}, \frac{3}{13}, \frac{3}{17}, \dots\right\} \qquad \text{(monotonically decreasing)} \\
(u_{n_k}) :=& \{u_2, u_4, u_6, \dots, u_{2k}, \dots\} \\
=& \left\{\frac{1}{4}, \frac{1}{16}, \dots, \frac{1}{4^k}, \dots\right\} \qquad \text{(monotonically decreasing)} \\
(v_{n_k}) :=& \{v_2, v_4, s_6, \dots, v_{2k}, \dots\} \\
=& \left\{\left(1+\frac{1}{2}\right), \left(1+\frac{1}{4}\right), \left(1+\frac{1}{6}\right), \dots, \left(1+\frac{1}{2k}\right), \dots\right\} \qquad \text{(monotonically decreasing)} \\
(w_{n_k}) :=& \{w_2, w_4, w_6, \dots, w_2k, \dots\} \\
=& \{4, 2^4, 2^6, 2^8, \dots\} \qquad \text{monotonic increasing} \\
(x_{n_k}) :=& \{x_1, x_3, x_5, \dots, x_{2k-1}, \dots\} \\
=& \left\{\frac{1}{5}, \frac{1}{5}, \dots\right\} \qquad \text{constant hence monotonic} \\
(y_{n_k}) :=& \{y_1, y_3, \dots, y_{2k-1}, \dots\} \\
=& \{0, 0, \dots\} \qquad \text{constant hence monotonic} \\
(z_{n_k}) :=& \{z_8, z_16, \dots, z_{8k}, \dots\} \\
=& \{8, 16, 32, \dots\} \qquad \text{monotonic increasing.}
\end{aligned}
$$

For each sequence, give the set S of subsequential limits.

$$\begin{aligned} s_n &: S = \left\{-1, -\frac{1}{2}, \frac{1}{2}, 1\right\} \\ t_n &: S = \{0\} \\ u_n &: S = \{0\} \\ v_n &: S = \{-1, 1\} \\ w_n &: S = \{\}, \qquad \text{the empty set} \\ x_n &: S = \left\{\frac{1}{5}, 5\right\} \\ y_n &: S = \{0, 2\} \\ z_n &: S = \{0\}. \end{aligned}$$

For each sequence, give the lim sup and lim inf.

$$\begin{aligned} &\liminf s_n = -1, \limsup s_n = 1 \\ &\liminf t_n = 0, \limsup t_n = 0 \\ &\liminf u_n = 0, \limsup u_n = 0 \\ &\liminf v_n = -1, \limsup v_n = 1 \\ &\liminf w_n = -\infty, \limsup w_n = +\infty \\ &\liminf x_n = \frac{1}{5}, \limsup x_n = 5 \\ &\liminf y_n = 0, \limsup y_n = 2 \\ &\liminf z_n = -\infty, \limsup z_n = 0. \end{aligned}$$

Which sequences converge? Which diverge to $-\infty$, which diverge to $+\infty$?

Sequences $(t_n), (u_n)$ converge, as we see:

$$\liminf t_n = \limsup t_n = \lim t_n$$
$$\liminf u_n = \limsup u_n = \lim u_n.$$

Which sequences are bounded?

All our sequences but $(w_n), (z_n)$ are bounded; that is, $(s_n), (t_n), (u_n), (v_n), (x_n), (y_n)$ are all bounded.

□

[11.5] Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.
Give the set of subsequential limits for (q_n).
Give the values of $\limsup q_n$ and $\liminf q_n$.

Solution. We define (q_n) as our enumeration of rationals in the interval $(0, 1]$, so our set S of subsequential limits for (q_n) should be infinite, and is:

$$S = \{x \in \mathbb{R} : 0 \leq x \leq 1\},$$

which we equivalently write as $[0, 1]$.

And thus we conclude:

$$\liminf q_n = 0, \qquad \limsup q_n = 1.$$

□

[11.9]
Show the closed interval $[a, b]$ is a closed set.

Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

Solution. To show the closed interval $[a, b]$ is a closed set, we just show that the operation lim is invariant over this interval. That is, $\lim s_n \in [a, b]$, where all $s_n \in [a, b]$. To see this explicitly, we cite **Problem 8.9c** from Homework 2, whose claims are included below.
Put simply, $(0, 1)$ is an open interval. The canonical counter-example to this is $s_n := \frac{1}{n}$, where $n = 1, 2, \dots$ has $(\lim s_n) \notin (0, 1)$. □

[8.9. from HW2] (whose result in part c we use here for Problem 11.9) Let (s_n) be a sequence that converges.

1. Show that if $a \leq s_n$ for all but finitely many n, then $a \leq \lim s_n$.
2. Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.
3. Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

[12.3] Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

$$(s_n) := (0,1,2,1,0,1,2,1,0,1,2,1,0,1,2,1,0,\dots)$$
$$(t_n) := (2,1,1,0,2,1,1,0,2,1,1,0,2,1,1,0,2,\dots).$$

Find the following values.

First we compile some useful information:

$$(s_n + t_n) := (2,2,3,1,2,2,3,1,\dots)$$
$$(s_n t_n) := (0,1,2,0,0,1,2,0,\dots),$$

and

$$\liminf s_n = 0, \qquad \limsup s_n = 2$$
$$\liminf t_n = 0, \qquad \limsup t_n = 2$$
$$\liminf(s_n + t_n) = 1, \qquad \limsup(s_n + t_n) = 3$$
$$\liminf(s_n t_n) = 0, \qquad \limsup(s_n t_n) = 2.$$

Solution. Now we can simply find the following desired quantities:

$$\liminf s_n + \liminf t_n = 0 + 0 = 0$$
$$\liminf(s_n + t_n) = 1$$
$$\liminf s_n + \limsup t_n = 0 + 2 = 2$$
$$\limsup(s_n + t_n) = 3$$
$$\limsup s_n + \limsup t_n = 2 + 2 = 4$$
$$\liminf(s_n t_n) = 0$$
$$\limsup(s_n t_n) = 2.$$

□

[12.4] Show $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). *Hint:* First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply exercise 9.9(c).

Solution. We suppose (s_n) and (t_n) are bounded. Consider then, from the triangle inequality, that:

$$|s_n + t_n| \leq |s_n| + |t_n|.$$

Because $(s_n), (t_n)$ are bounded, it is easy to see that $(s_n + t_n)$ is also bounded (take for instance the sum of their individual upper bounds as the new bound for the sum). Now because $(s_n), (t_n), (s_n + t_n)$ are bounded, by the completeness property, we have:

$$\sup\{|s_n + t_n|\} \leq \sup\{|s_n|\} + \sup\{|t_n|\}.$$

Because $(s_n), (t_n), (s_n + t_n)$ are bounded, the sequences $(|s_n|), (|t_n|), (|s_n + t_n|)$ must also be bounded. To see this explicitly, simply notice 0 is a lower bound and the max of $|\inf s_n|$ and $|\sup s_n|$ of each sequence is an upper bound for $(|s_n|)$ and similarly for the others. By a similar argument, we can show the reverse direction, and so we may drop the absolute values to get:

$$\sup\{s_n + t_n\} \leq \sup\{s_n\} + \sup\{t_n\}.$$

Now we can simply consider the tail-ends of each sequence for $n > N$ for some N across all our sequences $(s_n), (t_n), (s_n + t_n)$ simultaneously to have:

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\},$$

as suggested by the hint. Now we can use exercise 9.9(c), which claims (and we have proven) that if $\exists_{N_0} [s_n \leq t_n, \forall n > N_0]$ and $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$. Hence using the result of 9.9(c), we simply conclude that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n,$$

as desired.

□

We also look at the previous problem, 12.3, and verify that our result there is consistent with this proven claim here in 12.4.

[12.14] Calculate: (1) $\lim(n!)^{1/n}$ and (2) $\lim \frac{1}{n}(n!)^{1/n}$.

Solution. First we show $\lim \frac{1}{n}(n!)^{1/n} = \frac{1}{e}$. Suppose we know Stirling's Approximation (joking). It suffices to only use the 'capstone' theorem from this section, which states that for any sequence of **nonzero** real numbers, we have:

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf \left|[s_n]^{1/n}\right| \leq \limsup \left|[s_n]^{1/n}\right| \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|,$$

and its corollary: If $L := \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists, then $\lim |s_n|^{1/n} = L$ exists.
Let us define

$$s_n := \frac{n^n}{n!},$$

for which $n \in \mathbb{N}$ surely gives all strictly positive (nonzero) terms in the sequence. Then

$$s_{n+1} = \frac{(n+1)^n(n+1)}{(n+1)n!} = \frac{(n+1)^n}{n!},$$

and

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)^n}{n!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n,$$

which looks suspiciously like the "compound interest" definition of e. That is, in class, we discussed various ways of defining transcendental constants like e or π, and one such definition was:

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Applying the theorem, we get the following compound inequality:

$$\liminf \left(1 + \frac{1}{n}\right)^n \leq \liminf \left[\frac{n^n}{n!}\right]^{1/n} \leq \limsup \left[\frac{n^n}{n!}\right]^{1/n} \leq \limsup \left(1 + \frac{1}{n}\right)^n.$$

Recall that convergence of some s_n occurs if and only if $\liminf s_n = \limsup s_n = \lim s_n$; hence we have $\lim s_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = e$. Then from the corollary, we have:

$$\lim \left|\frac{n^n}{n!}\right|^{1/n} = e,$$

and hence by our limit theorems (sequence terms all nonzero),

$$\lim \left[\frac{1}{n}(n!)^{1/n}\right] = \lim \left[\left(\frac{n!}{n^n}\right)^{1/n}\right] = \frac{\lim 1}{\lim \left|\frac{n^n}{n!}\right|^{1/n}} = \frac{1}{e},$$

as required for **part (2)**.
For **part (1)**, we simply consider $a_n := n!$ so that $\left|\frac{a_{n+1}}{a_n}\right| = n + 1$. The left inequality in the above gives:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| = \liminf |n+1| = +\infty \leq \liminf \left|(n!)^{1/n}\right|,$$

so we have $\limsup \left|(n!)^{1/n}\right| \geq \liminf \left|(n!)^{1/n}\right| \geq +\infty$, from which we conclude:

$$\lim(n!)^{1/n} = +\infty.$$

□