Math 104: Homework 2

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

6.2 Show that if α and β are Dedekind cuts, then so is $\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

Solution: Let $\alpha, \beta \subset \mathbb{Q}$ be Dedekind cuts. To show $\alpha + \beta$ is a Dedekind cut, we must verify three properties. First, we claim $\alpha + \beta$ is a nonempty proper subset of $\mathbb{Q}$. To see this, note that each of α and β is nonempty, so we can find $r_1 \in \alpha$, $r_2 \in \beta$, meaning $r_1 + r_2 \in \alpha + \beta$, so $\alpha + \beta$ is nonempty. On the other hand, since α and β are Dedekind cuts we must have $\alpha \neq \mathbb{Q}$ and $\beta \neq \mathbb{Q}$, meaning we can find $M_1 \in \mathbb{Q}$ with $M_1 \notin \alpha$ and $M_2 \in \mathbb{Q}$ with $M_2 \notin \beta$. We claim that in fact $r_1 < M_1$ for all $r_1 \in \alpha$ and $r_2 < M_2$ for all $r_2 \in \beta$. This is because clearly we cannot have $r_1 = M_1$ or $r_2 = M_2$ by assumption that $M_1 \notin \alpha$ and $M_2 \notin \beta$. On the other hand, we also can't have $r_1 > M_1$ because then property (ii) of Dedekind cuts would require $M_1 \in \alpha$. Similarly we cannot have $r_2 > M_2$. So since $r_1 < M_1$ for all $r_1 \in \alpha$ and $r_2 < M_2$ for all $r_2 \in \beta$ we must have $r_1 + r_2 < M_1 + M_2$, and since r_1 and r_2 were arbitrary this shows that $s < M_1 + M_2$ for all $s \in \alpha + \beta$. In particular, this means $M_1 + M_2 \notin \alpha + \beta$, so $\alpha + \beta$ is indeed a proper subset of $\mathbb{Q}$.

The second property we must verify is that $\alpha + \beta$ is "closed downward," i.e. if $s_1 \in \alpha + \beta$ and $s_2 \in \mathbb{Q}$ with $s_2 < s_1$ then $s_2 \in \alpha + \beta$ as well. To see this, let $s_1 = r_1 + r_2 \in \alpha + \beta$ (i.e. with $r_1 \in \alpha$ and $r_2 \in \beta$) and let $s_2 \in \mathbb{Q}$ with $s_2 < s_1$. Then this means $s_2 < r_1 + r_2$ so $s_2 - r_2 < r_1$. Then since α is a Dedekind cut and therefore closed downward, we must have $s_2 - r_2 \in \alpha$. Then we must have $(s_2 - r_2) + r_2 \in \alpha + \beta$, so indeed $s_2 \in \alpha + \beta$ as desired.

The final property we must verify is that $\alpha + \beta$ has no maximal element. To see this, let $s_1 \in \alpha + \beta$ and we will show there exists some $s_2 \in \alpha + \beta$ with $s_2 > s_1$. This is because, if $s_1 = r_1 + r_2$ for some $r_1 \in \alpha$ and $r_2 \in \beta$ then since α has no maximal element we can find $r_1' \in \alpha$ with $r_1' > r_1$. Then if we take $s_2 = r_1' + r_2$ we indeed have $s_2 \in \alpha + \beta$ with $s_2 > s_1$, so $\alpha + \beta$ has no maximal element.

6.3 (a) Show $\alpha + 0^* = \alpha$ for all Dedekind cuts α.

Solution: Let $\alpha \subset \mathbb{Q}$ be an arbitary Dedekind cut. By definition, the Dedekind cut $0^* \subset \mathbb{Q}$ is the set consisting of all negative rational numbers. Now let $s \in \alpha + 0^*$, say $s = r_1 + r_2$ with $r_1 \in \alpha$ and $r_2 \in 0^*$. Then since $r_2 < 0$ we have $r_1 + r_2 < r_1$ so since α is closed downward we have $s = r_1 + r_2 \in \alpha$, showing that $\alpha + 0^* \subset \alpha$.

On the other hand, let $r_1 \in \alpha$. Then since α has no maximal element we can find $r_2 \in \alpha$ with $r_2 > r_1$. This means $r_1 - r_2 < 0$ and therefore $r_1 - r_2 \in 0^*$. So then $r_2 + (r_1 - r_2) \in \alpha + 0^*$, so $r_1 \in \alpha + 0^*$, meaning $\alpha \subset \alpha + 0^*$ as well. Since containment holds in both directions, we have the equality of sets $\alpha = \alpha + 0^*$.

(b) We claimed, without proof, that addition of Dedekind cuts satisfies property A4. Thus if α is a Dedekind cut, there is a Dedekind cut $-\alpha$ such that $\alpha + (-\alpha) = 0^*$. How would you define $-\alpha$?

Solution: The "naive" approach would be to attempt the definition $-\alpha = \{-r : r \in \alpha\}$. However, a quick sketch of what this would look like on the number line should be enough to convince one that this will not work, since Dedekind cuts are supposed to essentially resemble intervals of the form $(-\infty, x)$ and this definition would produce something of the form $(-x, \infty)$. However, there is a simple solution to this issue, since (again, as we can see intuitively from a sketch) the naive definition actually happens to produce what looks like the *complement* of the set we are aiming for – i.e. it produces the set of all rational numbers *greater* than the place where we are "cutting" the rational line. So we should expect the definition $-\alpha = \mathbb{Q} \setminus \{-r : r \in \alpha\} = \{r \in \mathbb{Q} : -r \notin \alpha\}$ to work. There is still one technical problem, however. If the original Dedekind cut α happened to represent a rational number, then its complement $\mathbb{Q} \setminus \alpha$ will have a minimum element (equal to the rational number α represents). Under negation, this will unfortunately become a maximal element of $-\alpha$. However, $-\alpha$ is easy to define in this case, because in this case we have $\alpha = s^* = \{r \in \mathbb{Q} : r < s\}$ for some $s \in \mathbb{Q}$, so we can just directly define $-\alpha = (-s)^* = \{r \in \mathbb{Q} : r < -s\}$.

While the problem does not require you to verify that this definition is satisfactory, I'll do so here anyway for your edification. Our first task is to show that our definition in fact produces a Dedekind cut. Clearly since we defined $-\alpha = (-s)^*$ in the case $\alpha = s^*$ for some $s \in \mathbb{Q}$, we have that $-\alpha$ is a Dedekind cut because we already know all sets of the form q^* for some $q \in \mathbb{Q}$ are Dedekind cuts, so we focus on the case where α is not rational.

To see that $-\alpha$ is nonempty, note that since α is not all of $\mathbb{Q}$ we can find $r \in \mathbb{Q}$ with $r \notin \alpha$. Then we see (since α is not rational), $-r \in -\alpha$ since $r \notin \alpha$. On the other hand, we cannot have $-\alpha = \mathbb{Q}$ because α is nonempty, and for any $r \in \alpha$ we have $-r \notin -\alpha$.

To see that $-\alpha$ is closed downward, let $r \in -\alpha$ and $s < r$. Then that means $-r \notin \alpha$, and we also have $-s > -r$. This means we cannot have $-s \in \alpha$ because then since α is closed downward that would mean $-r \in \alpha$ as well. So since $-s \notin \alpha$ we in fact have $s \in -\alpha$ as desired.

To see $-\alpha$ has no maximal element, suppose for the sake of contradiction that it does, say $s = \max(-\alpha)$. Then we claim $-s = \min(\mathbb{Q} \setminus \alpha)$. To see this, let $r \in \mathbb{Q} \setminus \alpha$. Then since $r \notin \alpha$ we have $-r \in -\alpha$, which means $-r < s$, meaning $r > -s$, so $-s$ is a lower bound of $\mathbb{Q} \setminus \alpha$. On the other hand we also have since $s \in -\alpha$ that $-s \notin \alpha$, meaning $-s \in \mathbb{Q} \setminus \alpha$, so indeed $-s = \min(\mathbb{Q} \setminus \alpha)$. However, this means s is the least upper bound for α, i.e. $s = \sup \alpha$, contradicting the assumption that α was not rational.

Now that we have verified that our definition of $-\alpha$ indeed produces a Dedekind cut, we still must verify that it satisfies the desired property of being the additive inverse of α, i.e. that $\alpha + (-\alpha) = 0^*$ as we have defined it. First we will handle the case $\alpha = s^*$. First let $r \in \alpha + (-\alpha) = s^* + (-s)^*$. Then we have $r = r_1 + r_2$ where $r_1 \in s^*$ and $r_2 \in (-s)^*$. So $r_1 < s$ and $r_2 < -s$. But then $r_1 + r_2 < s + -s = 0$, so $r_1 + r_2 \in 0^*$, showing $s^* + (-s)^* \subseteq 0^*$. On the other hand, let $r \in 0^*$. Then $s + r/2 < s$ and $-s + r/2 < -s$ so $s + r/2 \in s^*$ and $-s + r/2 \in (-s)^*$. So then $r = (s + r/2) + (-s + r/2) \in s^* + (-s)^*$. So we have shown $s^* + (-s)^* \subseteq 0^*$, completing the proof that $s^* + (-s)^* = 0^*$.

Now suppose α is irrational, and let $r \in \alpha + (-\alpha)$, meaning $r = r_1 + r_2$ where $r_1 \in \alpha$ and $r_2 \in -\alpha$. Then we claim $-r_2 > r_1$. This is because $r_2 \in -\alpha$ means $-r_2 \in \mathbb{Q} \setminus \alpha$, but if $-r_2 \leq r_1$ we would have $-r_2 \in \alpha$ since α is closed downward. So since $-r_2 > r_1$ we have $r_1 + r_2 < 0$, meaning $r = r_1 + r_2 \in 0^*$, showing $\alpha + -\alpha \subseteq 0^*$. On the other hand, let $r \in 0^*$. Now we claim we can find $s \in \alpha$ with $s - r/2 \notin \alpha$. This is because if we take an arbitrary $q \in \alpha$ we can consider the set $\{q - nr/2 : n \in \mathbb{N}\}$, which is unbounded above (by the Archimedean property) and therefore contains some element which is not in α since α itself is bounded above. Then by taking the smallest n for which $q - nr/2 \notin \alpha$ we see that $q - (n-1)r/2$ gives the desired choice of s. Now if we take $r_1 = s + r/2$ we clearly have $r_1 \in \alpha$ since $s + r/2 < s$. Moreover, if we take $r_2 = -(s - r/2) = -s + r/2$, we have $r_2 \in -\alpha$ since $-r_2 \notin \alpha$ by our choice of s. So then

$r_1 + r_2 = s + r/2 + -s + r/2 = r \in \alpha + (-\alpha)$, showing $0^* \subset \alpha + (-\alpha)$, completing the proof that $\alpha + (-\alpha) = 0^*$ in the case α is irrational.

7.3 For each sequence below, determine whether it converges and, if it converges, give its limit. No proofs are required.

(f) $s_n = 2^{1/n}$

Solution: The intuition here is that $\frac{1}{n} \to 0$, so $2^{1/n} \to 2^0 = 1$.

(i) $\frac{(-1)^n}{n}$

Solution: We have $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \to 0$, so we expect $\frac{(-1)^n}{n} \to 0$ as well.

(r) $\left(1 + \frac{1}{n}\right)^2$

Solution: Since $\frac{1}{n} \to 0$, we expect $\left(1 + \frac{1}{n}\right)^2 \to (1+0)^2 = 1$.

(q) $\frac{3^n}{n!}$

Solution: Though we haven't proved the "ratio test" for sequences yet, we can use it here because the problem does not require rigorous proof. We see that in this case $|s_{n+1}/s_n| = 3/(n+1) \to 0$, so we expect $3^n/n! \to 0$.

7.4 Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

Solution: We can take a sequence like $\sqrt{2}/n$, which converges to zero.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

Solution: This is a little bit trickier. The problem does not explicitly ask for a proof that the sequence you exhibit actually converges to an irrational number, so I think simply describing the example should suffice. In fact, there are many readily available to us. One straightforward example would be the sequence $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ whose n^{th} term is the truncation of the decimal expansion of $\sqrt{2}$ to the first n digits. Another famous example would be the sequence $(1 + 1/n)^n$ which converges to e, or the sequence defined recursively by $s_1 = 1$ and $s_{n+1} = (s_n + 2s_n^{-1})/2$ for $n \geq 1$. This sequence also converges to $\sqrt{2}$ (this is known as "Newton's Method" for approximating square roots – if the sequence is defined by $s_{n+1} = (s_n + as_n^{-1})/2$ then $s_n \to \sqrt{a}$, at least provided $a > 0$ and $s_1 > 0$ as well).

8.2 (e) Determine the limit of the sequence $s_n = \frac{1}{n}\sin n$ and prove your claim.

Solution: Our intuition is that, even though $\sin n$ behaves somewhat unpredictably, it is at least bounded between ± 1, so that asymptotically the growth of n in the denominator will render the oscillation of $\sin n$ irrelevant and the sequence should converge to zero. Indeed, showing the limit is zero entails choosing N so that $|(1/n)\sin n| < \varepsilon$ for $n > N$, and we immediately see that we can directly exploit the bound $|(1/n)\sin n| \leq |1/n| = 1/n$ (since $|\sin n| \leq 1$), and we have $1/n < \varepsilon$ so long as $n > 1/\varepsilon$. So our proof might go as follows:

Let $\varepsilon > 0$. Choose $N > 1/\varepsilon$. Then for all $n > N$ we have $n > 1/\varepsilon$, meaning $1/n < \varepsilon$. But then this means $|(1/n)\sin n| \leq 1/n < \varepsilon$ for all $n > N$, so we have shown that $(1/n)\sin n \to 0$.

8.5 (a) Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

Solution: Let (a_n), (b_n) and (s_n) be sequences as in the problem statement. As usual with general problems of this sort, we must exploit the convergence of (a_n) and (b_n) to control the behavior of (s_n). To do this, we consider the question of the manner in which the "error bounds" on a_n and b_n constrain s_n. The answer in this case (which is not difficult to see if one makes a rudimentary sketch of the situation) turns out to be particularly simple – if $|a_n - s| < \varepsilon$ and $|b_n - s| < \varepsilon$ then a_n and b_n are both in the interval $(s - \varepsilon, s + \varepsilon)$ and since s_n is between them then s_n must be as well. So if we let $\varepsilon > 0$ and simply choose N_1 so that $|a_n - s| < \varepsilon$ for $n > N_1$ and N_2 so that $|b_n - s| < \varepsilon$ for $n > N_2$, then taking $N = \max\{N_1, N_2\}$ guarantees that for $n > N$ we have both $|a_n - s| < \varepsilon$ and $|b_n - s| < \varepsilon$. In particular, this means $s - \varepsilon < a_n$ and $b_n < s + \varepsilon$, so we have $s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$, i.e. $s - \varepsilon < s_n < s + \varepsilon$, meaning $|s_n - s| < \varepsilon$ for all $n > N$ as well. This shows $s_n \to s$.

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

Solution: Note that we have $0 \leq |s_n| \leq t_n$, so, since $t_n \to 0$, applying the squeeze lemma from (a) with $a_n = 0$ and $b_n = t_n$ shows that $|s_n| \to 0$. Now, we also have that $-|s_n| \leq s_n \leq |s_n|$, so a second application of the squeeze lemma shows that $s_n \to 0$ as well. (Technically we are using the fact that $|s_n| \to 0$ implies $-|s_n| \to 0$ which we would either have to prove on its own or justify using a theorem from section 9, but let's not worry about that.)

8.8 (b) Prove that $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$. *Hint*: Try rewriting the sequence as

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \frac{n}{\sqrt{n^2+n} + n}.$$

Solution: As in the hint, we will work with the expression

$$s_n = \frac{n}{\sqrt{n^2+n} + n}.$$

This means we would like to show that with a suitable choice of N we can ensure $|s_n - (1/2)| < \varepsilon$ (for any $\varepsilon > 0$). With a little further manipulation, we see

$$\begin{aligned}
|s_n - s| &= \left| \frac{n}{\sqrt{n^2+n} + n} - \frac{1}{2} \right| = \left| \frac{2n - (\sqrt{n^2+n} + n)}{2(\sqrt{n^2+n} + n)} \right| \\
&= \left| \frac{n - \sqrt{n^2+n}}{2\sqrt{n^2+n} + 2n} \right| \\
&= \left| \frac{1 - \sqrt{1 + \frac{1}{n}}}{2\sqrt{1 + \frac{1}{n}} + 2} \right| \\
&= \frac{\sqrt{1 + \frac{1}{n}} - 1}{2 + 2\sqrt{1 + \frac{1}{n}}}.
\end{aligned}$$

To bound this last expression above by ε, it would be nice to first bound it above by some expression that doesn't involve radicals. The denominator can simply be bounded below by 4, since $\sqrt{1+1/n} > 1$. On the other hand, we also have $1 + 1/n > \sqrt{1+1/n}$ since $\sqrt{1+1/n} > 1$, so we see the numerator can be bounded above by $1 + 1/n - 1 = 1/n$. In all, this means we have $|s_n - s| < 1/(4n)$, so a choice like $N > 1/(4\varepsilon)$ should suffice, as then we will have $1/(4n) < \varepsilon$, which in turn guarantees $|s - s_n| < \varepsilon$ for $n > N$.

8.9 Let (s_n) be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

Solution: Suppose for the sake of contradiction that $s = \lim s_n < a$. Then pick $\varepsilon = (a - s)/2$. Since $s_n \to s$, we can find N such that $|s_n - s| < \varepsilon$ (for all $n > N$) for this choice of ε, meaning $s_n < s + (a - s)/2 = (a + s)/2 < (a + a)/2 = a$ since $s < a$. But since this is true for all $n > N$, we have shown that $s_n < a$ for infinitely many n, contradicting our assumption that $s_n \geq a$ for all but finitely many n.

(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

Solution: The argument is very similar to (a). If we suppose $s = \lim s_n > b$ then by taking $\varepsilon = (s - b)/2$ we ultimately find we can choose N so that $s_n > (s + b)/2 > b$ for all $n > N$, contradicting that $s_n \leq b$ for all but finitely many n.

(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

Solution: This follows by invoking both (a) and (b) above. The assumption that all but finitely many s_n belong to $[a, b]$ directly implies that all but finitely many s_n satisfy $a \leq s_n \leq b$, so by (a) the first inequality means $\lim s_n \geq a$ and by (b) the second implies $\lim s_n \leq b$, so since $a \leq \lim s_n \leq b$ we indeed have $\lim s_n \in [a, b]$.

9.3 Suppose $\lim a_n = a$ and $\lim b_n = b$, and $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

Solution: Since $\lim a_n = a$ and $\lim b_n = b$, we have by several applications of Theorem 9.4 that $\lim a_n^3 = a^3$ and $\lim b_n^2 = b^2$. We also have by 9.2 that $\lim 4a_n = 4a$, so using 9.3 we find $\lim(a_n^3 + 4a_n) = a^3 + 4a$. On the other hand clearly $\lim 1 = 1$ so by 9.3 again we have $\lim(b_n^2 + 1) = b^2 + 1$. Now clearly $b_n^2 + 1 > 0$ and $b^2 + 1 > 0$, so we have by 9.6 that

$$\lim \frac{a_n^3 + 4a_n}{b_n^2 + 1} = \frac{a^3 + 4a}{b^2 + 1},$$

as desired.

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = \infty$ then $\lim t_n = \infty$.

Solution: Suppose $\lim s_n = \infty$. Let $M \in \mathbb{R}$. Then we can find N_1 so that $s_n > M$ for $n > N_1$. Take $N = \max\{N_0, N_1\}$. Then for $n > N$ we have $t_n \geq s_n > M$. Since M was arbitrary, we see $t_n \to \infty$ as well.

(b) Prove that if $\lim t_n = -\infty$ then $\lim s_n = -\infty$.

Solution: Suppose $\lim t_n = -\infty$. Let $M \in \mathbb{R}$. Then we can find N_1 so that $t_n < M$ for $n > N_1$. Take $N = \max\{N_0, N_1\}$. Then for $n > N$ we have $s_n \leq t_n < M$. Since M was arbitrary, we see $s_n \to -\infty$ as well.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Solution: Suppose $\lim s_n = s$ and $\lim t_n = t$. Suppose for the sake of contradiction $t < s$. Then take $\varepsilon = (s - t)/2$ and choose N_1 so that $|s_n - s| < \varepsilon$ for $n > N_1$ and N_2

so $|t_n - t| < \varepsilon$ for $n > N_2$, and put $N = \max\{N_0, N_1, N_2\}$. Then for $n > N$ we have $s_n > s - \varepsilon = (s+t)/2$, while on the other hand $t_n < t + \varepsilon = (s+t)/2$, so we see that $t_n < s_n$ for $n > N$, contradicting the assumption that $s_n \leq t_n$ for $n > N_0$.

9.12 Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint:* Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

Solution: Let $\varepsilon > 0$. As in the hint, we choose a with $L < a < 1$. Then if we take $\varepsilon' = a - L$, we can use the assumption $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$ to find N for which

$$\left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < \varepsilon',$$

which means in particular we have

$$\left|\frac{s_{n+1}}{s_n}\right| < L + \varepsilon' = a,$$

and after multiplying on both sides by $|s_n|$ we see we indeed have $|s_{n+1}| < a|s_n|$ for $n > N$. We now argue by induction that this means $|s_n| < a^{n-N}|s_N|$ for $n > N$. To see this, note that the argument we just made shows the base case $|s_{N+1}| < a|s_N|$. Now suppose we know $|s_k| < a^{k-N}|s_N|$ for some particular $k > N$. Then we have $|s_{k+1}| < a|s_k|$ by our previous argument, and by the inductive hypothesis this means $|s_{k+1}| < a(a^{k-N}|s_N|) = a^{k+1-N}|s_N|$, completing the induction.

Now since we know $a^n \to 0$ (by Theorem 9.7(b)), we can find some N' for which $|a^n - 0| = a^n < \varepsilon/|s_N|$ for $n > N'$. Then if we take $N'' = N + N'$ we see that for $n > N''$ we have

$$|s_n| < a^{n-N}|s_N| < \frac{\varepsilon}{|s_N|}|s_N| = \varepsilon,$$

since for $n > N''$ we have $n - N > N'$.

(b) Show that if $L > 1$, then $\lim |s_n| = \infty$. *Hint:* Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

Solution: As in the hint, we let $t_n = \frac{1}{|s_n|}$. Then we see

$$\lim \left|\frac{t_{n+1}}{t_n}\right| = \lim \left|\frac{|s_{n+1}|^{-1}}{|s_n|^{-1}}\right| = \lim \left|\frac{s_{n+1}}{s_n}\right|^{-1}.$$

Since we know $\lim \left|\frac{s_{n+1}}{s_n}\right| = L > 0$, we can apply Lemma 9.5 to say

$$\lim \left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{L} < 1.$$

Then we see by part (a) that we must have $t_n \to 0$. But then by 9.10 (since $|s_n| > 0$ for all n) we see this implies $\lim |s_n| = \infty$.

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Solution: Say $s_n = a^n/n!$. Note that

$$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \left|\frac{a^{n+1}/(n+1)!}{a^n/n!}\right| = \lim \left|\frac{a}{n+1}\right| = 0,$$

so by part (a) of problem 9.12 we have that $\lim s_n = 0$ as well.