Math 104, Summer 2019

PSET #2 (due Tuesday 7/2/2019)

Problem 6.2. Show that if α and β are Dedekind cuts, then so is $\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

Solution. This result is explicitly given in the main text of Ross section 6 *as the definition of addition* in $\mathbb{R}$. However, we'll back-track and verify this result.
Suppose α, β are Dedekind cuts. Then we can explicitly write $\alpha = \{r_1 \in \mathbb{Q} \mid r_1 < s_1\}$ and $\beta = \{r_2 \in \mathbb{Q} \mid r_2 < s_2\}$. To show $(\alpha + \beta) := \{r_1 + r_2 \mid r_1 \in \alpha, r_2 \in \beta\}$ is a Dedekind cut, we simply refer to the definition of (requirements for) a Dedekind cut.

(i) To see $(\alpha + \beta) \neq \mathbb{Q}$, simply consider elements $\xi_1, \xi_2 \in \mathbb{Q}$ but with $\xi_1 \notin \alpha$ and $\xi_2 \notin \beta$. By our definitions, $(\xi_1 + \xi_2) \notin (\alpha + \beta)$, but $(\xi_1 + \xi_2) \in \mathbb{Q}$, hence $(\alpha + \beta) \neq \mathbb{Q}$. To see $(\alpha + \beta) \neq \{\}$, consider that α, β are nonempty hence r_1, r_2 exist, and thus $r_1 + r_2 \in (\alpha + \beta)$ exists. To see this inclusion explicitly, see (ii).
(ii) Let $r_1 \in \alpha, r_2 \in \beta$. Then we have for all $s_1, s_2 \in \mathbb{Q}$, that $(s_1 < r_1) \implies s_1 \in \alpha$ and $s_2 < r_2 \implies s_2 \in \beta$. Adding these inequalities, we have $[(s_1 + s_2) < (r_1 + r_2)] \implies (s_1 + s_2) \in (\alpha + \beta)$ per our definition of $(\alpha + \beta)$. Because $s_1 \in \mathbb{Q}, s_2 \in \mathbb{Q} \implies (s_1 + s_2) \in \mathbb{Q}$, we satisfy requirement (ii).
(iii) We already have that α, β each contain no largest rational. In other words, $\forall r_1 \in \alpha$ and $\forall r_2 \in \beta$, we have the existence of some c_1, c_2 with $r_1 < c_1$ and $r_2 < c_2$. Because these all exist, $c_1 + c_2 \in \mathbb{Q}$. Adding the inequalities, for all $(r_1 + r_2) \in (\alpha + \beta)$, we must have $(r_1 + r_2) < c_1 + c_2$, and hence there exists some $(c_1 + c_2) \in \mathbb{Q}$ larger than $(r_1 + r_2) \in (\alpha + \beta)$ for all choices $r_1 \in \alpha, r_2 \in \beta$. We conclude $(\alpha + \beta)$ has no greatest rational.

As we have satisfied all requirements from the definition, we conclude $(\alpha + \beta)$ is a Dedekind cut.

□

Problem 6.3.
1. Show $\alpha + 0^* = \alpha$ for all Dedekind cuts α.
2. We claimed, without proof, that addition of Dedekind cuts satisfies property A4. Thus if α is a Dedekind cut, there is a Dedekind cut $-\alpha$ such that $\alpha + (-\alpha) = 0^*$. How would you define $-\alpha$?

Solution. **(1)** Recall that we defined $\alpha \leq \beta \iff \alpha \subseteq \beta$. Hence to show $\alpha = \beta$, we choose to show $\alpha \subseteq \beta$ and $\alpha \subseteq \beta$ (as opposed to the equivalent inequality argument).
We define 0^* in the natural way: $0^* := \{r_0 \in \mathbb{Q} \mid r_0 < 0\}$, or in other words the set of all negative rationals, where 0^* corresponds to the real number 0.
Let $r_1 \in \alpha$ and $r_0 \in 0^*$. From above, we have $r_2 < 0$. Hence $r_1 + r_0 < r_1$, and by our definition of α, $(r_1 + r_0) \in \alpha$. Thus from above, we have $\alpha + 0^* \subseteq \alpha$.
Let $r_1, r_2 \in \alpha$ with $r_1 < r_2$. Then $r_1 - r_2 < 0 \implies (r_1 - r_2) \in 0^*$. Then add (using the result of Problem 6.2 above) $r_2 \in \alpha$ and $(r_1 - r_2) \in 0^*$ to get: $r_2 + (r_1 - r_2) = r_1 \in \alpha$, so $\alpha + 0^* \subseteq \alpha$.
We have shown $\alpha \subseteq \alpha + 0^* \subseteq \alpha$, so we necessarily have $\alpha + 0^* = \alpha$ as required.

(2) Now we need to define some Dedekind cut $-\alpha$ so that $\alpha + (-\alpha) = 0^*$. As in Rudin, we propose

$$(-\alpha) := \{\ p \mid \exists_{r>0}\ [-p - r \notin \alpha]\ \}.$$

That is, some rational number less than $-p$ is not in α. Although not prompted, we prove this claim To show this, we first show $(-\alpha) \in \mathbb{R}$ as a Dedekind cut and then $\alpha + (-\alpha) = 0^*$.
(i) If $s \notin \alpha$ and $p = -s - 1$, then $-p - 1 \notin \alpha$, and hence $p \in (-\alpha)$. So $(-\alpha)$ is non-empty. If $q \in \alpha$, then $-q \notin (-\alpha)$, so $(-\alpha) \neq \mathbb{Q}$.
(ii) Fix $p \in (-\alpha)$ and $r > 0$, so $-p - r \notin \alpha$. Such an example exists by (i) above as we showed $(-\alpha)$ nonempty. Consider $(q < p) \implies (-q - r > -p - r) \implies -q - r \notin \alpha \implies q \in (-\alpha)$. We have shown that $q < p \implies q \in (-\alpha)$, as required.
(iii) Take any $p \in (-\alpha), r > 0$ as above. Fix $t := p + \frac{r}{2}$. Then $t > p$, and $-t - \frac{r}{2} = -p - r \notin \alpha$, so $t \in (-\alpha)$. But $t = p + \frac{r}{2} > p$, and this holds true for all $p \in (-\alpha)$, so $(-\alpha)$ has no maximal element.

Thus we conclude $(-\alpha) \in \mathbb{R}$; that is, it is a Dedekind cut. To see $\alpha + (-\alpha) \subset 0^*$, consider that

$$(r \in \alpha, s \in (-\alpha) \implies (-s \notin \alpha) \implies (r < -s) \implies (r + s < 0).$$

To see inclusion in the other direction ($\alpha+\beta \supset 0^*$), let $v \in 0^*$ and set $w := \frac{-v}{2}$. Then $w > 0$, and by the Archimedean property of $\mathbb{Q}$, there is an integer n with $nw \in \alpha$ but with $(n+1)w \notin \alpha$. Define $p := -(n+2)w$. Then surely, $(-p - w \notin \alpha) \implies p \in (-\alpha)$. Moreover,

$$v = nw + p \in \alpha + (-\alpha).$$

Hence $\alpha + (-\alpha) \supset 0^*$. Because $0^* \subset \alpha + (-\alpha) \subset 0^*$, we conclude that $\alpha + (-\alpha) = 0^*$, which validates our proposition. □

Problem 7.3. For each sequence below, determine whether it converges and, if it converges, give its limit. **No proofs are required.**

(f) $s_n = 2^{1/n}$

(i) $\frac{(-1)^n}{n}$

(r) $\left(1 + \frac{1}{n}\right)^2$

(q) $\frac{3^n}{n!}$

Solution. If we provide the limit, it follows by definition that the sequence converges.

(f) $\lim 2^{1/n} = 1$

(i) $\lim \frac{(-1)^n}{n} = 1$

(r) Does not converge as $1/n$ is known not to converge: $\left(1 + \frac{1}{n}\right)^2 = \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

(q) $\frac{3^n}{n!} \approx \frac{3^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$, so we deduce $\lim \frac{3^n}{n!} = 0$. □

Problem 7.4. Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

Solution. The problem does not ask for proof or verification, so we simply state our examples and their limits.

(a) Very simply, we see that the sequence $x_n := \frac{\sqrt{n}}{n^2}$, surely a sequence of rational numbers, has a limit of (converges to) 0.

(b) A classical example of this is the ratio between fibonacci numbers, where $F_0 := 0, F_1 := 1, F_n := F_{n-1} + F_{n-2}$. Let us define $r_n := \frac{F_{n+1}}{F_n}$ for $n > 0$. It is a known fact that $\lim r_n = \frac{1+\sqrt{5}}{2}$, the golden ratio. Surely, it is a sequence of rational numbers, as it is the ratio between natural numbers; and we know $r_n \notin \mathbb{Q}$.

□

Problem 8.2. (e) Determine the limit of the sequence $s_n = \frac{1}{n} \sin n$ and prove your claim.

Solution. Suppose we do not know that $\lim a_n b_n = \lim a_n \lim b_n$ and that we are allergic to this fact.
Define $s_n := \frac{1}{n} \sin n$. Let $\epsilon > 0$ and $N := \frac{2}{\epsilon}$. We will show $s := \lim s_n = 0$. Consider that $(n > N)$ by definition of N gives $n > \frac{2}{\epsilon}$ which results in:

$$\begin{aligned} n > \frac{2}{\epsilon} = \frac{1}{\epsilon} \cdot 2 > \frac{1}{\epsilon} \cdot |\sin n| \quad & (2 > 1 \geq |\sin n|,\ \forall n) \\ \implies \epsilon > \frac{1}{n} \cdot |\sin n| \quad & (\epsilon > 0 \implies n > N > 0) \\ \implies \epsilon > \frac{|\sin n|}{|n|} = \left|\frac{1}{n} \sin n - 0\right|. & \end{aligned}$$

As we have shown $(n > \frac{2}{\epsilon}) \implies \left(\left|\frac{1}{n^2} \sin n - 0\right| < \epsilon\right)$, we thus have $\lim \frac{1}{n} \sin n = 0$, as desired.

□

Problem 8.5.

1. Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \le s_n \le b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."
2. Suppose (s_n) and (t_n) are sequences such that $|s_n| \le t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

Solution. **(1.)** Fix $\epsilon > 0$. To show $\lim s_n = s$, it suffices to show for our given ϵ that for some n,

$$s - \epsilon < s_n < s + \epsilon \implies |s_n - s| < \epsilon$$

at which point we have that s_n converges to s as desired.
Because we are given $\lim a_n = \lim b_n = s$, by definition of limit and convergence, we have some N_a with $(n > N_a) \implies (s - \epsilon < a_n)$ and likewise, we have some N_b with $(n > N_b) \implies (b_n < s + \epsilon)$, where we drop the absolute values because our hypothesis ensures $\forall n$ that $a_n \le s_n \le b_n$.
Because we have a finite amount (2) of conditions, we can simply assert

$$(n > \max\{N_a, N_b\}) \implies (s - \epsilon < a_n \le s_n \le b_n < s + \epsilon),$$

where the middle inequality is given in the hypothesis, and the strict inequalities at the sides are from the fact s_a, s_b converge to s. Subtracting s_n across the inequality, we get $-\epsilon < s_n - s < +\epsilon$ which is equivalent to $0 \ge |s_n - s| < \epsilon$, precisely the definition that gives $\lim_{n\to\infty} s_n = \lim s_n = s$, and we are done.

□

Solution. **(2.)** Given $(s_n), (t_n)$ sequences with $|s_n| \le t_n$ is equivalent to $-t_n \le s_n \le t_n$ for all n. Given $\lim t_n = 0$, it easily follows that $\lim -t_n = 0$, and the squeeze lemma (above) asserts that $\lim s_n = 0$ (for free). □

Problem 8.8. (b) Prove that $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$.

Solution. First we notice:

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \frac{n}{\sqrt{n^2+n} + n}.$$

Fix $\epsilon > 0$ and let $N := \left[\frac{1}{\left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 - 1}\right]$. Then $n > N$ implies:

$$\begin{aligned}
n > \left[\frac{1}{\left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 - 1}\right] &\implies \left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 - 1 > \frac{1}{n} \implies \left(\frac{1+2\epsilon}{1-2\epsilon}\right)^2 > \left(1 + \frac{1}{n}\right) \\
&\implies \frac{2 - (1-2\epsilon)}{1-2\epsilon} > \sqrt{1 + \frac{1}{n}} \\
&\implies \frac{2}{1-2\epsilon} > \sqrt{1 + \frac{1}{n}} + 1 = \sqrt{\frac{n^2}{n^2} + \frac{n}{n^2}} + \frac{n}{n} \\
&\implies \frac{2}{1-2\epsilon} > \frac{\sqrt{n^2+n} + n}{n} \\
&\implies \frac{1-2\epsilon}{2} = \left(\frac{1}{2} - \epsilon\right) < \left[\frac{n}{\sqrt{n^2+n} + n}\right] \\
&\implies -\epsilon < \left(\sqrt{n^2+n} - \frac{1}{2}\right) \\
&\implies \left|(\sqrt{n^2+n} - n) - \frac{1}{2}\right| < \epsilon,
\end{aligned}$$

as required to show $\lim\left[\sqrt{n^2+n} - n\right] = \frac{1}{2}$.

□

Problem 8.9. Let (s_n) be a sequence that converges.

1. Show that if $a \leq s_n$ for all but finitely many n, then $a \leq \lim s_n$.
2. Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.
3. Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

Solution. (1.) We want to show that $(a \leq s_n)$ for all but finitely many n implies $a \leq \lim s_n$. Because there is only a finite number of n where $a > s_n$, it is easy to see there exists $N_1 \in \mathbb{N}$ with $(n > N_1) \implies (s_n \geq a)$. Because s_n converges, let $s := \lim s_n$.
Suppose for contradiction that $s < a$. Then we can fix $\epsilon := a - s > 0$ and choose some $N \geq N_1$. Then because $\lim s_n = s$, we have:

$$\begin{aligned}(n > N) &\implies |s_n - s| < \epsilon = a - s \\ &\iff |s_n - s| < (a - s) \\ &\implies -(a - s) < (s_n - s) < (a - s)\end{aligned}$$

Where the right inequality $s_n - s < a - s$ implies $s_n < a$ for all $n > N$, which contradicts our hypothesis that $a \leq s_n$ for all but finitely many n. Then it must be so that our supposition was wrong that $s < a$, and instead, we have $s \geq a$ as required.

(2.) Now we want to show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$. We proceed analogously as in part (1) above. Because s_n converges, let $s := \lim s_n$. Because there is only a finite number of n where $s_n > b$, it is easy to see there exists $N_2 \in \mathbb{N}$ with $(n > N_2) \implies (s_n \leq b)$.
Suppose for contradiction that $s > b$. Then we can fix $\epsilon := s - b > 0$ and choose some $N \geq N_2$. Then because $\lim s_n = s$, we have:

$$\begin{aligned}(n > N) &\implies |s_n - s| < \epsilon = s - b \\ &\iff |s_n - s| < s - b \\ &\iff -(s - b) = b - s < (s_n - s) < s - b \\ &\implies b < s_n,\end{aligned}$$

hence for all $n > N$, we have $b < s_n$, but this is a contradiction to the given statement that the set of such n is finite. Hence our supposition $s < b$ is false, and we must have $s \geq b$, as desired.
(3.) Suppose that all but a finite number of s_n belong to $[a, b]$. Let $U \subset \{n\}$ be this finite subset of n for which $s_n \notin [a, b]$. Surely, U satisfies the hypotheses of (1) and (2); that is, because only a finite many n have the property: $(a > s_n) \cup (s_n < b)$, parts (1) and (2) above assert that $a \leq \lim s_n \leq b$. Hence

$$\lim s_n \in [a, b],$$

which was to be shown. □

Problem 9.3. Suppose $\lim a_n = a$ and $\lim b_n = b$, and $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

Solution. We have $a := \lim a_n$, $b := \lim b_n$. Consider:

$$\begin{aligned}
\lim s_n &= \lim \frac{a_n^3 + 4a_n}{b_n^2 + 1} \\
&= \frac{\lim \left[a_n^3 + 4a_n\right]}{\lim \left[b_n^2 + 1\right]} \quad \text{(Ross Thm 9.6)} \\
&= \frac{\lim \left(a_n^3\right) + \lim \left[4a_n\right]}{\lim \left(b_n^2\right) + \lim (1)} \quad \text{(Ross Thm 9.3)} \\
&= \frac{\left(\lim a_n\right)^3 + \lim (4) \cdot \lim (a_n)}{\left(\lim b_n\right)^2 + \lim (1)} \quad \text{(Ross Thm 9.4, exponentiation as iterated mult.)} \\
&= \frac{a^3 + (\lim 4)a}{b^3 + (\lim 1)} \quad \text{(substituting known quantities } a, b.) \\
&= \frac{a^3 + 4a}{b^2 + 1}, \quad (\lim 4 = 4, \lim 1 = 1)
\end{aligned}$$

precisely as required.

□

Problem 9.9. Suppose there exists N_0 such that $s_n \le t_n$ for all $n > N_0$.

1. Prove that if $\lim s_n = \infty$ then $\lim t_n = \infty$.

   Solution. Fix some $M > 0$. Because $\lim s_n = +\infty$, we have the existence of some $N \ge N_0$ with $s_n > M, \forall n > N$. Then for all $n > N \ge N_0$, we have $M < s_n \le t_n$, which shows $\lim t_n = +\infty$. □

2. Prove that if $\lim t_n = -\infty$ then $\lim s_n = -\infty$.

   Solution. Like in (a), fix some $M < 0$. Because $\lim t_n = -\infty$, we have some $N \ge N_0$ with $t_n < M, \forall n > N$. Then for all $n > N \ge N_0$, we have $s_n \le t_n < M$, which shows $\lim s_n = -\infty$. □

3. Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \le \lim t_n$.

   Solution. We are given the existence of some N_0 past which, for all $n > N_0$, we have $s_n \le t_n$. If $\lim t_n = -\infty$, we have proven the desired result in (2). If $\lim s_n = \infty$, we have proven this in (1). Surely, we cannot have t_n converging but $s_n \to +\infty$, as this would contradict $s_n \le t_n$ in the hypothesis. Correspondingly, s_n cannot converge if $t_n \to -\infty$, as seen by the symmetric argument. Because we are given s, t exist as defined above, it only remains to prove $\lim s_n \le \lim t_n$ where $\lim s, \lim t \ne \pm\infty$.

   We have shown in 8.9 part (1) (letting $a := 0$) that $t_n \ge s_n \implies (t_n - s_n) \ge 0$ for all $n > N_0$ (with only finite points $n \le N_0$ otherwise) implies $\lim(t_n - s_n) \ge 0$. And by the limit theorems, this implies $\lim s_n \le \lim t_n \ge$, as desired. □

Problem 9.12. Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

1. Show that if $L < 1$, then $\lim s_n = 0$. *Hint:* Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.
2. Show that if $L > 1$, then $\lim |s_n| = \infty$. *Hint:* Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

Solution. **(1)** Notice that because we do not know if $|s_n|, |s_{n+1}|$ themselves converge, we cannot apply the usual limit theorem. We proceed as suggested by the hint. Suppose $L := \lim \left|\frac{s_{n+1}}{s_n}\right| < 1$. Because L is a limit of absolute values and all $s_n \neq 0$, we have $0 < L$.

Fix a with $0 < L < a < 1$, and let $\epsilon := (a - L) > 0$. Then because L exists, there exists some N for which

$$
\begin{aligned}
(n \geq N) \implies & \left|\frac{|s_{n+1}|}{|s_n|} - L\right| < \epsilon = (a - L) \\
\iff -(a - L) < & \frac{|s_{n+1}}{s_n} - L < a - L \iff 2L - a < \frac{|s_{n+1}|}{|s_n|} < a \\
\iff & (2L - a)|s_n| < |s_{n+1}| < a|s_n|,
\end{aligned}
$$

where the right inequality holds for all $n \geq N$.

We claim

$$|s_n| < a^{n-N}|s_N|$$

for all $n > N$ and show this formally by induction (although redundant because convergence guarantees all $n > N$). Consider the subset $U \subset \mathbb{N}$ of indicies i for which we have $|s_{N+i}| < a^i|s_N|$. We have shown $|s_{N+1}| < a|s_N|$ above, so $1 \in U$. Assume $k \in U$ so that $|s_{N+k}| < a^k|s_N|$. Recall that $0 < L < a < 1$ and that convergence to L guarantees that

$$(n > N) \implies |s_{n+1}| < a|s_N|.$$

So letting $n := N + k + 1 > N$, we immediately have $|s_{N+(k+1)}| < a|s_{N+k}|$. Using consequences of ordered field axioms and $k \in U$, we have

$$(|s_{N+k+1}|)(|s_{N+k}|) < (a|s_{N+k}|)\left(a^k|s_N|\right).$$

Because $s_n > 0$ for all n, we can divide across the inequality to get

$$|s_{N+k+1}| < a^k|s_N|,$$

and we have $k + 1 \in U$. Because we showed $k \in U \implies k + 1 \in U$, we have $U = \mathbb{N}$ by the induction axiom. Hence we have $|s_n| < a^{n-N}|s_N|$ as desired, for all $n > N$.

Then from the above, we have $0 < \frac{|s_n|}{|s_N|} < a^{n-N}$, for all $n > N$. Consider that

$$\lim a^{n-N} = \frac{\lim a^n}{\lim a^N},$$

as the denominator is constant and the numerator is known to converge to 0 for $0 < a < 1$. Then by the limit theorems, we have

$$\lim 0 = 0 \leq \lim \frac{|s_n|}{|s_N|} \leq \lim a^{n-N} = 0.$$

By the squeeze lemma we have proven, this asserts that $\lim \frac{|s_n|}{|s_N|} = 0$. Because the denominator is constant as N is fixed, we conclude that

$$\lim |s_n| = 0 \implies \lim s_n = 0,$$

which was to be shown. □

Solution. **(2)** Now we show that if $L > 1$, then $\lim |s_n| = \infty$. Per custom, we follow the provided hint and let $t_n := \frac{1}{|s_n|}$. Let $R := 1/L$. Because $L > 1$, then $0 < R < 1$. That is,

$$0 < \frac{1}{\lim \left|\frac{s_{n+1}}{s_n}\right|} = \lim \frac{|s_n|}{|s_{n+1}|} = \lim \frac{|t_{n+1}|}{|t_n|} < 1.$$

By our definitions, $\lim \frac{|t_{n+1}|}{|t_n|} = R$. Fix some $0 < a < R < 1$. Using R instead of L and t_n instead of s_n, part (i) above gives us that there exists some N with $(n > N) \implies \left(0 < \frac{|s_n|}{|s_N|} = \frac{|t_N|}{|t_n|} < a^{n-N}\right)$. Because these are all positive, by consequences of axioms of an ordered field, we have the following for the reciprocals:

$$\forall n > N, \qquad \frac{|t_n|}{|t_N|} > a^{N-n}$$
$$\frac{|t_n|}{|t_N|} > \frac{a^N}{a^n}$$

Because for $n > N$, $\frac{|t_n|}{|t_N|}$ is greater than $\frac{a^N}{a^n}$, and it is known that for all $n > N$, $0 < a < 1 \implies \frac{1}{a^n} > 1$, then we conclude

$$\lim \frac{a^N}{a^n} = a^N \lim \frac{1}{a^n} = a^N \cdot (+\infty) = +\infty,$$

which was to be shown.

□

Problem 9.15. Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Solution. Using our result in 9.12 above, we can see that $s_n := \frac{a^n}{n!} \neq 0$ for all n. Moreover, we have that

$$L := \lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \left|\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}\right| = \lim \left|\frac{a}{n+1}\right| = 0 < 1 \implies \lim s_n = 0,$$

by the limit theorems, precisely as required.

□