Math 104: Homework 1

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

1.1 Prove $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Solution: We argue by induction. When $n = 1$ we see that the sum on the left is simply 1, while the formula on the right gives $(1/6)(1)(2)(3) = 1$ as well, completing the "base case." For the inductive step, we assume $1^2 + 2^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ for some fixed (but unknown) $k \in \mathbb{N}$. Adding $(k+1)^2$ to both sides, we find

$$\begin{aligned} 1^2 + \ldots + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right] \\ &= \frac{1}{6}(k+1)\Big[k(2k+1) + 6(k+1)\Big] \\ &= \frac{1}{6}(k+1)\Big[2k^2 + 7k + 6\Big] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3), \end{aligned}$$

showing that the proposed formula holds for $n = k+1$ as well, thus completing the inductive step.

1.12 For $n \in \mathbb{N}$, let $n!$ [read "n factorial"] denote the product $1 \cdot 2 \cdot 3 \cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The *binomial theorem* asserts that

$$\begin{aligned} (a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \ldots + nab^{n-1} + b^n. \end{aligned}$$

(a) Verify the binomial theorem for $n = 1$, 2 and 3.

Solution: When $n = 1$, the left side is simply $a+b$ while the right is $\binom{1}{0}a + \binom{1}{1}b = a+b$, as desired. For $n = 2$, the left is $(a+b)^2 = a^2 + 2ab + b^2$ while the right is $\binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$ as well. For $n = 3$, the left is $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ while the right is $\binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$, and computing these binomial coefficients shows the two expressions agree again in this case.

(b) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

Solution: Applying the definition to the left-hand side of the desired identity gives

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!\Big[(k-1)!(n-k+1)! + k!(n-k)!\Big]}{k!(n-k)!(k-1)!(n-k+1)!}$$

Note that both terms in brackets in the numerator on the right have a common factor of $(k-1)!(n-k)!$ since $k! = k(k-1)!$ and $(n-k+1)! = (n-k+1)(n-k)!$. By factoring this out of the numerator and cancelling with the corresponding factors on the denominator we find

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!\Big[(n-k+1)+k\Big]}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!} \\ &= \frac{(n+1)!}{k!((n+1)-k)!} \\ &= \binom{n+1}{k},\end{aligned}$$

as desired.

(c) Prove the binomial theorem using mathematical induction and part (b).

Solution: By part (a), we need only do the inductive step. Suppose

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \ldots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

for some $k \in \mathbb{N}$. Multiplying both sides of this equation by $a+b$ we find

$$\begin{aligned}(a+b)^{k+1} &= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \ldots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] \\ &= \binom{k}{0}a^k(a+b) + \binom{k}{1}a^{k-1}b(a+b) + \ldots + \binom{k}{k-1}ab^{k-1}(a+b) + \binom{k}{k}b^k(a+b) \\ &= \binom{k}{0}a^{k+1} + \binom{k}{0}a^k b + \binom{k}{1}a^k b + \binom{k}{1}a^{k-1}b^2 + \ldots \\ &\quad + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k-1}ab^k + \binom{k}{k}ab^k + \binom{k}{k}b^{k+1} \\ &= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \ldots \\ &\quad + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1} \\ &= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \ldots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1},\end{aligned}$$

where we obtained the last line from the penultimate by using the result of (b) as well as the fact that $\binom{k}{0} = 1 = \binom{k+1}{0}$ and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$. The expression in the last line can be recognized as what the binomial theorem predicts for the expansion of $(a+b)^n$ when $n = k+1$, meaning we have completed the inductive step.

2.5 Show $[3+\sqrt{2}]^{2/3}$ is not a rational number.

Solution: Set $a = [3+\sqrt{2}]^{2/3}$. Then $a^3 = [3+\sqrt{2}]^2 = 11 + 6\sqrt{2}$, meaning $(a^3 - 11)^2 = 72$, so that a is a solution of the equation $x^6 - 22x^3 + 49 = 0$. By the rational zeros theorem, the only rational solutions this equation can have are $\pm 1, \pm 7$ and ± 49. For $x = \pm 1$ we find

$$\begin{aligned}1^6 - 22\cdot 1^3 + 49 &= 28 \qquad \text{and} \\ (-1)^6 - 22(-1)^3 + 49 &= 72,\end{aligned}$$

meaning ± 1 are not roots. When $x = \pm 7$ or ± 49 we can factor 49 out of the entire expression to see that the value of $x^6 - 22x^3 + 49$ is a product of 49 and an integer which has remainder 1 when divided by 7 and therefore cannot be zero in any of these cases either. Altogether this means there are no rational solutions to the equation $x^6 - 22x^3 + 49 = 0$, and since a is a solution, we conclude that a cannot be rational.

2.7 Show the following irrational-looking expressions are actually rational numbers:

(a) $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$

Solution: While it is possible to find a polynomial with integer coefficients which has this number as a root, it is challenging (though I will show how to do it at the end), and there's an easier way. Showing that $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ is rational is equivalent to showing that $\sqrt{4 + 2\sqrt{3}} = r + \sqrt{3}$ for some rational r. By squaring both sides, we see that this is equivalent to finding r such that $4 + 2\sqrt{3} = (r^2 + 3) + 2r\sqrt{3}$. Intuitively, for this to be possible, we expect that it must in fact be the case that $2r = 2$ and $r^2 + 3 = 4$, from which we immediately see that $r = 1$ is a candidate. Indeed, we have $(1 + \sqrt{3})^2 = 4 + 2\sqrt{3}$, meaning $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$, i.e. $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = 1$, which is indeed rational.

If we set $a = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$, now that we know $a = 1$ we could of course say something like "a is a root of the polynomial $x - 1$" or "a is a root of the polynomial $x^2 - 2x + 1$" or any number of things, but as humans we could only arrive at those statements in hindsight after realizing that in fact $a = 1$. If we imagine we didn't know this and wanted to find some polynomial of which a is a root from the *given expression* for a, we might proceed as follows: we see that

$$\begin{aligned} a^2 &= (4 + 2\sqrt{3}) - 2\sqrt{3}\sqrt{4 + 2\sqrt{3}} + 3 \\ &= 7 + 2\sqrt{3}\left(1 - \sqrt{4 + 2\sqrt{3}}\right), \end{aligned}$$

meaning

$$\begin{aligned} \left(a^2 - 7\right)^2 &= 12\left(1 - \sqrt{4 + 2\sqrt{3}}\right)^2 \\ &= 12\left(1 - 2\sqrt{4 + 2\sqrt{3}} + 4 + 2\sqrt{3}\right) \\ &= 12\left(5 - 2\sqrt{4 + 2\sqrt{3}} + 2\sqrt{3}\right) \\ &= 12\left(5 - 2\left(\sqrt{4 + 2\sqrt{3}} - \sqrt{3}\right)\right). \end{aligned}$$

The trick here (as you might see from the suggestive phrasing of the last line above) is to recognize that a itself has reoccured in the expression on the right, and by exploiting this fact we can write

$$(a^2 - 7)^2 = 12(5 - 2a).$$

From here one could, in principle, use the rational zeros theorem to generate a list of candidate rational values for a (among which one would find the true value $a = 1$) but I think it's already clear that this isn't exactly the most efficient approach to the problem.

(b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

Solution: Proceeding as in the first method for (a), we write $\sqrt{6+4\sqrt{2}} = r+\sqrt{2}$ and square both sides to find $6+4\sqrt{2} = (r^2+2)+2r\sqrt{2}$. We should expect $r^2+2=6$ and $2r=4$ in this case, and indeed $r=2$ causes both of these equations to be satisfied, and one can verify that in fact $6+4\sqrt{2} = (2+\sqrt{2})^2$, so that $\sqrt{6+4\sqrt{2}} = 2+\sqrt{2}$, meaning $\sqrt{6+4\sqrt{2}}-\sqrt{2} = 2$, which is, of course, rational.

3.3 Prove (iv) and (v) of Theorem 3.1 [in the textbook – page 15].

Solution: For (iv), we can argue as follows: by Theorem 3.1 part (iii) we have $(-a)(-b) = -(a(-b))$. By A2 (commutativity), we have $-(a(-b)) = -((-b)a)$, and applying (iii) again tells us that the latter expression is equal to $-(-ba)$, which after another appeal to A2 is equal to $-(-ab)$. As a lemma, we can show that $-(-c) = c$ for any c, and this will complete our argument. To show this, we can argue that on one hand $-(-c) + -c = 0$ by A4, but on the other hand $-c+c=0$ again by A4, meaning $-(-c)+-c = -c+c$. Then (i) of Theorem 3.1 tells us we can cancel $-c$ to find $-(-c) = c$. Applying this to our former situation tells us $-(-ab) = ab$, completing the proof that $(-a)(-b) = ab$.

For (v), we note that if $ac = bc$ and $c \neq 0$ then by M4 (multiplicative inverses) there must exist c^{-1}, so multiplying both sides by c^{-1} tells us $(ac)c^{-1} = (bc)c^{-1}$, and by two applications of M1 (associativity) this is equivalent to $a(cc^{-1}) = b(cc^{-1})$. By M4 we have $cc^{-1} = 1$, so $a \cdot 1 = b \cdot 1$. Then by M3 (multiplicative identity) this is the same as $a = b$, completing the argument.

3.4 Prove (v) and (vii) of Theorem 3.2 [in the textbook – page 16].

Solution: For (v): note that by M3 (multiplicative identity) we have $1^2 = 1$, so by (iv) of Theorem 3.2 we must have $0 \leq 1^2 = 1$. Since $0 \neq 1$, this means indeed $0 < 1$.

For (vii): Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then since $0 \leq a \leq b$ by O3 (transitivity) we have $0 \leq b$. Now we cannot have $0 = b$ because this would mean $0 \leq a \leq 0$ which would imply by O2 (antisymmetry) that $a = 0$, contradicting the assumption $0 < a$. So we must have $0 < b$ as well. [Note that we can now conclude a^{-1} and b^{-1} exist in the first place since both a and b are nonzero.] By applying Theorem 3.2 (vi) to both of these inequalities we find $0 < a^{-1}$ and $0 < b^{-1}$. By Theorem 3.2 (iii) we then have $0 \leq a^{-1}b^{-1}$, and by applying O5 to the inequality $a < b$ with $c = a^{-1}b^{-1}$ we see $a(a^{-1}b^{-1}) < b(a^{-1}b^{-1})$. Applying M2 (commutativity) gives $a(a^{-1}b^{-1}) < b(b^{-1}a^{-1})$, and then applying M1 (associativity) to both sides gives $(aa^{-1})b^{-1} < (bb^{-1})a^{-1}$. By M4 we have $1 \cdot b^{-1} < 1 \cdot a^{-1}$ and by M3 (multiplicative identity) this gives $b^{-1} < a^{-1}$ as desired. We already noted that $0 < b^{-1}$, so we are done.

3.6 (a) Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$. *Hint*: Apply the triangle inequality twice. Do *not* consider eight cases.

Solution: By the triangle inequality, $|(a+b)+c| \leq |a+b|+|c|$, and again by the triangle inequality $|a+b| \leq |a|+|b|$. Adding $|c|$ to both sides of the latter inequality gives $|a+b|+|c| \leq |a|+|b|+|c|$, so chaining this with the previous gives us $|a+b+c| \leq |a+b|+|c| \leq |a|+|b|+|c|$ so the result follows by transitivity.

(b) Use induction to prove

$$|a_1+a_2+\ldots+a_n| \leq |a_1|+|a_2|+\ldots+|a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

Solution: The inequality is trivial when $n = 1$ and when $n = 2$ it is just the triangle inequality. For the inductive step, suppose that for some $k \in \mathbb{N}$ we have $|a_1 + \ldots + a_k| \leq |a_1| + \ldots + |a_k|$ for all choices of k numbers $a_1, \ldots, a_k \in \mathbb{R}$. Then let $a_{k+1} \in \mathbb{R}$. Adding $|a_{k+1}|$ to both sides of this inequality gives $|a_1 + \ldots + a_k| + |a_{k+1}| \leq |a_1| + \ldots + |a_{k+1}|$. By the ordinary triangle inequality, we also have $|a_1 + \ldots + a_{k+1}| \leq |a_1 + \ldots + a_k| + |a_{k+1}|$, so the inequality $|a_1 + \ldots + a_{k+1}| \leq |a_1| + \ldots + |a_{k+1}|$ follows by transitivity. Since a_{k+1} was arbitrary, we have completed the inductive step and hence the proof.

3.7 (a) Show $|b| < a$ if and only if $-a < b < a$.

Solution: Suppose $|b| < a$. If $b \geq 0$, then $|b| = b$ so $b < a$. Also, $a > 0$ so $-a < 0 \leq b$ so indeed $-a < b < a$. On the other hand, if $b < 0$, we have $|b| = -b$, so $-b < a$, meaning $-a < b$. Also, $a > 0 > b$ so in all we have $-a < b < a$ in this case too.

(b) Show $|a - b| < c$ if and only if $b - c < a < b + c$.

Solution: By part (a), $|a - b| < c$ implies $-c < a - b < c$. Adding b to this chain of inequalities gives the desired chain $b - c < a < b + c$.

4.3 For each set below, give its supremum if it has one. Otherwise write "NO sup."

(d) $\{\pi, e\}$

Solution: $\sup\{\pi, e\} = \pi$ and $\inf\{\pi, e\} = e$.

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Solution: $\sup\{\frac{1}{n} : n \in \mathbb{N}\} = 1$ and $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.

(i) $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$

Solution: The given set is actually equivalent to the interval $[0, 1]$. To see this, note that $[0, 1] \subset [-\frac{1}{n}, 1 + \frac{1}{n}]$ for all $n \in \mathbb{N}$, meaning $[0, 1] \subset \cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$. If $x > 1$ then we can find some n so that $1 + 1/n < x$, meaning $x \notin [-\frac{1}{n}, 1 + \frac{1}{n}]$. Similarly, if $x < 0$ we can find n so that $x < -1/n$ meaning again $x \notin [-\frac{1}{n}, 1 + \frac{1}{n}]$. Together, this shows that there are no elements in the intersection $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$ that are not in $[0, 1]$. All of this considered, we see the supremum of the intersection is simply 1 (which also happens to be the maximum), and the infimum is 0 (which is also the minimum).

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

Solution: It is not too difficult to see that $n + \frac{(-1)^n}{n}$ grows unboundedly as $n \to \infty$, so the set has no supremum (more "rigorously," for $n > 1$ we have $1/n < 1$ meaning $n + \frac{(-1)^n}{n} \geq n - \frac{1}{n} > n - 1$, and clearly the latter grows without bound). On the other hand, we have $n + \frac{(-1)^n}{n} = 0$ when $n = 1$ and by the previous argument all remaining elements must be positive, so the infimum (and minimum) is 0.

(r) $\cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$

Solution: The given set actually consists solely of the element 1 (the argument is similar to the previous part), so the supremum and infimum are both 1.

(u) $\{x^2 : x \in \mathbb{R}\}$

Solution: Clearly this set is not bounded above and therefore has no supremum (though with the definitions in section 5 we would say the supremum is ∞). The minimum value achieved by x^2 for $x \in \mathbb{R}$ is 0 when $x = 0$ so the infimum (and minimum) is 1.

4.4 Repeat exercise 4.3 for infima [plural of infimum].

Solution: See above.

4.9 Complete the proof that $\inf S = -\sup(-S)$ in Corollary 4.5 by proving (1) and (2) [in the textbook – page 23].

Solution: (1) is the assertion that $-s_0 \le s$ for all $s \in S$, where $s_0 = \sup(-S)$. For any $s \in S$, we have $-s \in -S$ so $-s \le s_0$ by definition. Negating both sides shows $s \ge -s_0$, as desired.

(2) is the assertion that if $t \le s$ for all $s \in S$ then $t \le -s_0$. Suppose $t \le s$ for all $s \in S$. Then $-t \ge -s$ for all $s \in S$ and since every element of $-S$ is of this form we see $-t$ is an upper bound for $-S$. This means $-t \ge s_0$, so $t \le -s_0$ as desired.

4.12 Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers*. Prove if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. *Hint*: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset \mathbb{I}$.

Solution: As suggested by the hint, we will first note that if $r \in \mathbb{Q}$ then $r + \sqrt{2} \in \mathbb{I}$. This is because if $r + \sqrt{2} = r'$ for some $r' \in \mathbb{Q}$ we would have $\sqrt{2} = r' - r$, contradicting that $\sqrt{2} \in \mathbb{I}$ since we must have $r' - r \in \mathbb{Q}$. Now let $a, b \in \mathbb{R}$ with $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$ and by the denseness of $\mathbb{Q}$ we can find $r \in \mathbb{Q}$ with $a - \sqrt{2} < r < b - \sqrt{2}$. Adding $\sqrt{2}$ to this we find $a < r + \sqrt{2} < b$. Taking $x = r + \sqrt{2}$ exhibits the desired inequality with $x \in \mathbb{I}$ as noted earlier.

4.14 Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) Prove $\sup(A+B) = \sup A + \sup B$. *Hint*: To show $\sup A + \sup B \le \sup(A+B)$, show that for each $b \in B$, $\sup(A + B) - b$ is an upper bound for A, hence $\sup A \le \sup(A + B) - b$. Then show $\sup(A + B) - \sup A$ is an upper bound for B.

Solution: Proceeding as in the hint, let $b \in B$. We claim $\sup(A + B) - b$ is an upper bound for A. To see this, let $a \in A$. Then $a + b \in A + B$ so $a + b \le \sup(A + B)$. Subtracting b from both sides we see $a \le \sup(A + B) - b$. Since a was arbitrary, we indeed see $\sup(A + B) - b$ is an upper bound for A. Then since $\sup A$ is the least upper bound for A we must have $\sup A \le \sup(A+B)-b$. This inequality can also be rearranged to show $b \le \sup(A+B) - \sup A$, and since b was arbitrary in B to begin with this shows $\sup(A+B) - \sup A$ is an upper bound for B, which means $\sup B \le \sup(A+B) - \sup A$. So then $\sup A + \sup B \le \sup(A + B)$ as desired.

On the other hand, note that $\sup A + \sup B$ is clearly an upper bound for $A + B$ because for any $c \in A + B$, say $c = a + b$, then $a \le \sup A$ and $b \le \sup B$ so $c = a + b \le \sup A + \sup B$. Since this shows $\sup A + \sup B$ is an upper bound for $A + B$ we must have $\sup(A + B) \le \sup A + \sup B$ as well. Combining the inequalities in both directions shows that in fact $\sup(A + B) = \sup A + \sup B$.

(b) Prove $\inf(A + B) = \inf A + \inf B$.

Solution: Taking $-A$, $-B$ and $-(A + B)$ defined as in the proof of Corollary 4.5, we first claim that $-(A+B) = -A+-B$. To see this, let $c =\in -(A+B)$. Then $c = -(a+b)$ for some $a \in A$ and $b \in B$, so $c = -a + -b$, meaning $c \in -A + -B$ as well. On the other hand if $c \in -A + -B$ then $c = -a + -b$ for some $a \in -A$ and $b \in -B$, so $c = -(a + b)$ and $a + b \in A + B$ meaning $c \in -(A + B)$.

Using this, we note that $\inf(A + B) = -\sup(-(A + B)) = -\sup(-A + -B)$ and by part (a) this is $-(\sup(-A) + \sup(-B)) = -\sup(-A) + -\sup(-B)$. Then using that $-\sup(-A) = \inf(A)$ and $-\sup(-B) = \inf B$ we have that the latter expression is $\inf A + \inf B$, as desired.

5.4 Show that $\inf S = -\sup(-S)$ in the case where S is not bounded below. [Note that you completed the proof of this statement when S is bounded below for problem 4.9 above.]

Solution: Let $S \subset \mathbb{R}$ be unbounded below. Since $\inf S = -\infty$, the problem boils down to showing that $\sup(-S) = \infty$, as then the equation will be shown true. For this, we let $M \in \mathbb{R}$. Since S is unbounded below, we must be able to find $s \in S$ with $s < -M$. Then we have that $-s > M$, so we have exhibited an element of $-S$ greater than M. Since M was arbitrary, this means $-S$ must be unbounded above, so $\sup(-S) = \infty$, as desired.