# Math 104: Homework 11

Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

30.3 Find the following limits if they exist.

(a) $\lim_{x\to\infty} \frac{x-\sin x}{x}$
**Solution:** Here we have (for $x > 0$, say)

$$\frac{x-\sin x}{x} = 1 - \frac{\sin x}{x},$$

and we see that clearly $\lim_{x\to\infty} 1 = 1$ and $\lim_{x\to\infty}(\sin x)/x = 0$ (since $|(\sin x)/x| \leq 1/x$ for $x > 0$). So then

$$\lim_{x\to\infty} \frac{x-\sin x}{x} = \lim_{x\to\infty} 1 + \lim_{x\to\infty} -\frac{\sin x}{x} = 1 + 0 = 1.$$

(b) $\lim_{x\to\infty} x^{\sin(1/x)}$
**Solution:** Appealing to exercise 30.4 below, we can argue that $\lim_{x\to\infty} x^{\sin(1/x)}$ exists if and only if the limit $\lim_{y\to 0^+}(1/y)^{\sin y}$ exists, and in case it does the values are equal. So let's analyze $\lim_{y\to 0^+}(1/y)^{\sin y}$. We can write the expression in the limit as $e^{\log(1/y)\sin y} = e^{-\log y \sin y}$. So let's set $g(y) = \log y \sin y$ for $y > 0$ and calculate $\lim_{y\to 0^+} g(y)$. We see that we can write

$$g(y) = \frac{\log y}{1/\sin y},$$

so take $h_1(y) = \log y$ and $h_2(y) = 1/\sin y$. Then as $y \to 0^+$ we have $\sin y \to 0^+$ so we clearly have $|h_2(y)| \to \infty$. Now consider the limit as $y \to 0^+$ of $h_1'/h_2'$. For $y > 0$ we have

$$\frac{h_1'(y)}{h_2'(y)} = \frac{1/y}{-(\cos y)(\sin y)^{-2}} = \frac{-(\sin y)^2}{y\cos y}.$$

Now consider the limit $\lim_{y\to 0^+}(\sin y)/y$. We clearly have that the numerator and denominator both approach zero, and taking derivatives of both yields the limit $\lim_{y\to 0^+}\cos y = 1$, so we see by L'Hospital's rule that $\lim_{y\to 0^+}(\sin y)/y = 1$. On the other hand we have $\lim_{y\to 0^+} -\sin y/\cos y = 0$ because $\sin y \to 0$ while $\cos y \to 1$. So in all we have

$$\lim_{y\to 0^+}\frac{h_1'(y)}{h_2'(y)} = \lim_{y\to 0^+}\frac{\sin y}{y}\cdot\frac{-\sin y}{\cos y} = \left(\lim_{y\to 0^+}\frac{\sin y}{y}\right)\left(\lim_{y\to 0^+}\frac{-\sin y}{\cos y}\right) = 1\cdot 0 = 0.$$

Then by another application of L'Hospital's rule, since we again clearly have $\lim_{y\to 0^+}|h_2(y)| = \infty$ we see that

$$\lim_{y\to 0^+} g(y) = \lim_{y\to 0^+}\frac{h_1(y)}{h_2(y)} = \lim_{y\to 0^+}\frac{h_1'(y)}{h_2'(y)} = 0.$$

So then we have that $\lim_{y\to 0^+} e^{-g(y)} = e^0 = 1$, which then implies that the original limit $\lim_{x\to\infty} x^{\sin(1/x)} = 1$ as well.

30.4 Let $f$ be a function defined on some interval $(0, a)$, and define $g(y) = f(\frac{1}{y})$ for $y \in (a^{-1}, \infty)$; here we set $a^{-1} = 0$ if $a = \infty$. Show $\lim_{x \to 0^+} f(x)$ exists if and only if $\lim_{y\to\infty} g(y)$ exists, in which case these limits are equal.

**Solution:** First suppose $\lim_{x\to 0^+} f(x)$ exists, say $\lim_{x \to 0^+} f(x) = L$. First consider the case $|L| < \infty$. Then let $\varepsilon > 0$. Then we can find $\delta > 0$ so that for $0 < x < \delta$ we have $|f(x)-L| < \varepsilon$. Then take $\alpha = 1/\delta$. We claim that for $y > \alpha$ we have $|g(y) - L| < \varepsilon$ as well. This is because if $y > \alpha$ then $0 < 1/y < 1/\alpha = \delta$, so by our choice of $\delta$ we have $|f(1/y) - L| < \varepsilon$. But by definition this means $|g(y) - L| < \varepsilon$, so indeed $\lim_{y\to\infty} g(y) = L$ in this case. Now suppose $L = \infty$ (the case $L = -\infty$ is almost identical). Then let $M > 0$. By assumption we can take $\delta > 0$ so that for $0 < x < \delta$ we have $f(x) > M$. Now set $\alpha = 1/\delta$. Then for $y > \alpha$ we again have $0 < 1/y < \delta$ so $f(1/y) > M$ which means directly that $g(y) > M$, showing that $\lim_{y\to\infty} g(y) = \infty$.

The proof in the other direction (assuming $\lim_{y\to\infty} g(y)$ exists) can be obtained by reversal of the above steps – i.e. by first choosing $\alpha$ so that the appropriate condition holds for $g$, and then taking $\delta = 1/\alpha$, so that the same condition holds for $f$.

30.7 The requirement in Theorem 30.2 [page 242 in the book] that $g'(x) \neq 0$ for $x$ "near" $s$ is important. In a careless application of L'Hospital's rule in which the zeros of $g'$ "cancel" the zeros of $f'$, erroneous results can be obtained. For $x \in \mathbb{R}$, let

$$f(x) = x + \cos x \sin x \quad \text{and} \quad g(x) = e^{\sin x}(x + \cos x \sin x).$$

(a) Show $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$.
**Solution:** In general we have $\cos x \sin x \geq -1$ so we see that $f(x) \geq x - 1$. It is straightforward that $\lim_{x\to\infty} x - 1 = \infty$, so we also have $\lim_{x\to\infty} f(x) = \infty$. Similarly, we see that $e^{\sin x} \geq e^{-1}$, so that $g(x) \geq e^{-1}(x - 1)$, and clearly $\lim_{x\to\infty} e^{-1}(x - 1) = \infty$ as well, so $\lim_{x\to\infty} g(x) = \infty$.

(b) Show $f'(x) = 2(\cos x)^2$ and $g'(x) = e^{\sin x} \cos x[2 \cos x + f(x)]$.
**Solution:** Since $x$, $\cos x$ and $\sin x$ are all differentiable on $\mathbb{R}$, the rules of differentiation show us that

$$f'(x) = 1+(-\sin x)(\sin x)+(\cos x)(\cos x) = (\cos x)^2+(\sin x)^2+(\cos x)^2-(\sin x)^2 = 2(\cos x)^2.$$

On the other hand since $\sin x$ is differentiable everywhere and $e^x$ is differentiable everywhere we have by the chain rule that the derivative is $(\cos x)e^{\sin x}$. Then because $g(x) = e^{\sin x} f(x)$ we see that

$$\begin{aligned} g'(x) &= (\cos x)e^{\sin x} f(x) + e^{\sin x} f'(x) \\ &= e^{\sin x}[(\cos x)f(x) + 2(\cos x)^2] \\ &= e^{\sin x}(\cos x)[2 \cos x + f(x)], \end{aligned}$$

as desired.

(c) Show $\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x} \cos x}{2 \cos x+f(x)}$ if $\cos x \neq 0$ and $x > 3$.
**Solution:** Clearly we have that $e^{\sin x} \neq 0$ always, and we claim that if $x > 3$ then $2 \cos x + f(x) \neq 0$ as well. This is because $2 \cos x + f(x) = x + (\cos x)(2 + \sin x)$, and we have that $(\cos x)(2 + \sin x) \geq -3$ since $|\cos x| \leq 1$ and $|2 + \sin x| \leq 3$, so if $x > 3$ then

indeed $2\cos x + f(x) > 0$. So we see that, if $x > 3$, we have that $g'(x) = 0$ exactly when $\cos x = 0$. So if $\cos x \neq 0$ then direct calculation shows

$$\frac{f'(x)}{g'(x)} = \frac{2(\cos x)^2}{e^{\sin x}(\cos x)[2\cos x + f(x)]} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)},$$

as desired.

(d) Show $\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0$ and yet the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does *not* exist.

**Solution:** We see that $|e^{-\sin x}| \leq e$ and $|\cos x| \leq 1$ so the expression $2e^{-\sin x}\cos x$ is bounded between $\pm 2e$. On the other hand, since $\lim_{x\to\infty} f(x) = \infty$ we clearly have $\lim_{x\to\infty} 2\cos x + f(x) = \infty$ since $2\cos x + f(x) \geq f(x) - 2$. So given $\varepsilon > 0$, choose $\alpha > 0$ so that $|f(x) + 2\cos x| > 2e/\varepsilon$ for $x > \alpha$, then we see that for $x > \alpha$ we have

$$\left|\frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}\right| \leq \frac{2e}{|2\cos x + f(x)|} < \frac{2e}{2e/\varepsilon} = \varepsilon.$$

On the other hand we clearly have for $x > 1$ that $f(x)/g(x) = e^{-\sin x}$ which fails to converge as $x \to \infty$ because it repeats the values $e$ and $e^{-1}$ infinitely many times and for arbitrarily large $x$ (specifically when $x = \pi/2 + n\pi$ for $n \in \mathbb{N}$).

31.1 Find the Taylor series for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

**Solution:** Take $f(x) = \cos x$. Then we have that $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$ and $f^{(4)}(x) = \cos x$ again, so that the expressions for the derivatives of $f$ repeat in cycles of four. Putting $x = 0$ in all of these, we see that if $n$ is odd then $f^{(n)}(x) = 0$, while if $n = 2k$ for some nonnegative integer $k$ we have $f^{(n)}(x) = (-1)^k$. Then using the formula $a_n = f^{(n)}(0)/n!$ we see the Taylor series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Now note that from our earlier calculation and the fact that $\sin x$ and $\cos x$ are both bounded in magnitude by 1, we see that $|f^{(n)}(x)| \leq 1$ for all $n \geq 0$ and $x \in \mathbb{R}$, which means we can apply Corollary 31.4 (where $a = -\infty$ and $b = \infty$) to conclude that $R_n(x) \to 0$ for all $x \in \mathbb{R}$, which is equivalent to saying the Taylor series converges to $\cos x$ for every $x \in \mathbb{R}$.

31.4 Consider $a, b$ in $\mathbb{R}$ where $a < b$. Show there exists infinitely differentiable functions $f_a$, $g_b$, $h_{a,b}$ and $h^*_{a,b}$ on $\mathbb{R}$ with the following properties. You may assume, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. The same applies to the quotient provided the denominator never vanishes.

(a) $f_a(x) = 0$ for $x \leq a$ and $f_a(x) > 0$ for $x > a$. *Hint*: Let $f_a(x) = f(x - a)$ where $f$ is the function in Example 3 [page 257 in the book].

**Solution:** As in the hint, we define $f_a(x) = f(x-a)$ for $x \in \mathbb{R}$. Then note that $f_a = f \circ g$ where $g(x) = x - a$. Then clearly $g$ is infinitely differentiable, and we saw in Example 3 that $f$ was infinitely differentiable, so it follows that $f_a$ is infinitely differentiable. Moreover, we know that when $x \leq a$ we have $x - a \leq 0$ so $f_a(x) = f(x - a) = 0$ because $f(y) = 0$ for $y \leq 0$. Similarly $f_a(x) = f(x - a) > 0$ when $x > a$ because here $x - a > 0$ and $f(y) > 0$ for $y > 0$.

(b) $g_b(x) = 0$ for $x \geq b$ and $g_b(x) > 0$ for $x < b$.

**Solution:** Consider $g_b(x) = f(b-x)$. Then arguments similar to those in part (a) show that $g_b$ has the desired properties.

(c) $h_{a,b}(x) > 0$ for $x \in (a,b)$ and $h_{a,b}(x) = 0$ for $x \notin (a,b)$.

**Solution:** Consider $h_{a,b}(x) = f_a(x)g_b(x)$. Then since $f_a$ and $g_b$ are infinitely differentiable we see that $h_{a,b}$ is infinitely differentiable. Also, if $\leq a$ we have $f_a(x) = 0$ so $h_{a,b}(x) = 0$, and similarly if $x \geq b$ we have $g_b(x) = 0$ so $h_{a,b}(x) = 0$ here as well. On the other hand if $x \in (a,b)$ we see that both $f_a(x) > 0$ and $g_b(x) > 0$ so $h_{a,b}(x) > 0$ for these values of $x$, as desired.

(d) $h^*_{a,b}(x) = 0$ for $x \leq a$ and $h^*_{a,b}(x) = 1$ for $x \geq b$. *Hint*: Use the function $\frac{f_a}{f_a+g_b}$.

**Solution:** As in the hint, take

$$h^*_{a,b}(x) = \frac{f_a(x)}{f_a(x) + g_b(x)}.$$

Now note first of all that $f_a(x) + g_b(x) > 0$ for all $x$, because $f_a$ and $g_b$ are both nonnegative everywhere, so $f_a(x) + g_b(x)$ would only be zero if $f_a(x) = g_b(x) = 0$, which would require $x \leq a$ and $x \geq b$ simultaneously which is impossible. So $h^*_{a,b}$ is defined everywhere and since quotients of infinitely differentiable functions are infinitely differentiable whereever the denominator is nonvanishing we see that $h^*_{a,b}$ is infinitely differentiable as well. Now consider $x \leq a$. Then we have $f_a(x) = 0$ which directly means $h^*_{a,b}(x) = 0$. On the other hand, suppose $x \geq b$. Then we see that $g_b(x) = 0$, so that $h^*_{a,b}(x) = f_a(x)/f_a(x) = 1$, as desired.

32.2 Let $f(x) = x$ for rational $x$ and $f(x) = 0$ for irrational $x$.

(a) Calculate the upper and lower Darboux integrals for $f$ on the interval $[0,b]$.

**Solution:** It is fairly clear that $L(f) = 0$ on $[0,b]$, because if $P$ is any partition of $[0,b]$ then we can always find some irrational $x \in [t_{k-1}, t_k]$ for each $k = 1, 2, \ldots, n$ for which $f(x) = 0$, showing that $m(f, [t_{k-1}, t_k]) = 0$ for each $k$, so that $L(f,P) = 0$ for the arbitrary partition $P$.

On the other hand, we claim that $U(f) = b^2/2$. To see this, first let $P$ be an arbitrary partition of $[0,b]$. First notice that $M(f, [t_{k-1}, t_k]) = t_k$. This is because we can always find rational numbers $x \in [t_{k-1}, t_k]$ which are arbitrarily close to $t_k$, for which $f(x) = x$ (while of course we also trivially have $f(x) \leq t_k$ on $[t_{k-1}, t_k]$). So the upper sum for $P$ is

$$U(f,P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})$$

Now notice that if we take $g(x) = x$ for all $x \in [0,b]$ we see that for any partition $P$ we have $U(f,P) = U(g,P)$, which straightforwardly implies that $U(f) = U(g)$. We claim $g$ is integrable with $\int_0^b g = b^2/2$.

To see this, suppose $P_n$ is the partition $\{0, b/n, 2b/n, \ldots, (n-1)b/n, b\}$. For this $P$ we have

$$U(g, P_n) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^{n} k = \frac{b^2}{2} \frac{n(n+1)}{n^2} = \frac{b^2}{2}\left(1 + \frac{1}{n}\right).$$

On the other hand, we have

$$\begin{aligned} L(g,P_n) &= \sum_{k=1}^{n} \frac{(k-1)b}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^{n} (k-1) \\ &= \frac{b^2}{n^2} \sum_{k=0}^{n-1} k = \frac{b^2}{2} \frac{n(n-1)}{n^2} = \frac{b^2}{2} \left(1 - \frac{1}{n}\right), \end{aligned}$$

so that

$$U(g,P_n) - L(g,P_n) = \frac{b^2}{n},$$

meaning we can clearly make $U(g,P_n) - L(g,P_n) < \varepsilon$ for sufficiently large $n$, showing by Theorem 32.5 that $U(g) = L(g)$. Also from the formulas we clearly see that $U(g,P_n) \to b^2/2$ and $L(g,P_n) \to b^2/2$, so $\int_0^b g = b^2/2$, as claimed. From the earlier analysis, this means that $U(f) = U(g) = b^2/2$ as well.

(b) Is $f$ integrable on $[0,b]$?

**Solution:** No, because $U(f) > L(f)$ on $[0,b]$.

32.5 Use Exercise 4.8 [page 27 in the book] to prove Theorem 32.4. Specify the sets $S$ and $T$ in this case.

**Solution:** Let

$$S = \{L(f,P) \,:\, P \text{ is a partition of } [a,b]\}$$

and

$$T = \{L(f,Q) \,:\, Q \text{ is a partition of } [a,b]\}.$$

Then by Lemma 32.3 we see that $s \le t$ for any $s \in S$ and $t \in T$. This means by Exercise 4.8 that $\sup S \le \inf T$, which by definition means $L(f) \le U(f)$ because $L(f) = \sup S$ and $U(f) = \inf T$.