Math 104, Summer 2019

PSET #11 (due Monday 8/12/2019)

Ross 30.2. L'Hospital's Rule. Let s signify one of $a \in \mathbb{R}, a^+, a^-, \infty, -\infty$, and suppose f, g are differentiable functions for which the following limit exists:

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L.$$

If $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$ or if $\lim_{x \to s} |g(x)| = +\infty$, then:

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L.$$

Problem 30.3. Find the following limits if they exist.

1. $\lim_{x \to \infty} \frac{x - \sin x}{x}$
2. $\lim_{x \to \infty} x^{\sin(1/x)}$

Solution. (1) Observe that $\lim_{x \to \infty} \frac{(x - \sin x)'}{x'}$ does not exist ($\lim_{x \to \infty} \cos x$ does not exist), so L'Hospital's does not apply in the present form. Instead, we write:

$$\lim_{x \to \infty} \frac{x - \sin x}{x} = \lim_{x \to \infty} 1 - \frac{\sin x}{x} = 1 - 0 = \boxed{1},$$

where we reach this conclusion by the fact that the numerator is bounded by $|\sin x| \leq 1$ as the denominator explodes (if vanishing is a legal term, then so is exploding) to $+\infty$.

(2) Directly, we can abuse the convention that $x^0 := 1$ for all $x \in \mathbb{R}$. Now because $\lim_{x \to \infty} \frac{1}{x} = 0$ and via composition of continuous functions being continuous, we have

$$\lim_{x \to \infty} x^{\sin(1/x)} = x^{\lim_{x \to \infty} \sin(1/x)} = x^{\sin(\lim_{x \to \infty} 1/x)} = x^{\sin 0} = x^0 = \boxed{1}.$$

□

Problem 30.4. Let f be a function defined on some interval $(0, a)$, and define $g(y) = f(\frac{1}{y})$ for $y \in (a^{-1}, \infty)$; here we set $a^{-1} = 0$ if $a = \infty$. Show $\lim_{x \to 0^+} f(x)$ exists if and only if $\lim_{y \to \infty} g(y)$ exists, in which case these limits are equal.

Solution. We first note that we defined $g(y) := f\left(\frac{1}{y}\right)$ for $y \in (a^{-1}, \infty)$, where $a > 0$ as a consequence of $(0, a)$ being an interval. Now we notice $y \in (a^{-1}, \infty)$ gives $\frac{1}{y} \in (0, a)$, so we conclude $g\left(\frac{1}{y}\right) = f(y)$.

($\Longrightarrow$) Suppose $K_1 := \lim_{x \to 0^+} f(x)$ exists, with $x \in (0, a)$. Then $\frac{1}{x} \in (a^{-1}, \infty)$, and by the above, we have:

$$K_1 = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} g\left(\frac{1}{x}\right) = \lim_{\frac{1}{x} \to +\infty} g(x),$$

as desired.

($\Longleftarrow$) Now suppose $K_2 := \lim_{y \to \infty} g(y)$ exists, with $y \in (a^{-1}, \infty)$. Then $\frac{1}{y} \in (0, a)$ and by the above, we have:

$$K_2 = \lim_{y \to \infty} g(y) = \lim_{y \to \infty} f\left(\frac{1}{y}\right) = \lim_{\frac{1}{y} \to 0^+} f(y),$$

as required.

□

Problem 30.7. The requirement in Theorem 30.2 [page 242 in the book] that $g'(x) \neq 0$ for x "near" s is important. In a careless application of L'Hospital's rule in which the zeros of g' "cancel" the zeros of f', erroneous results can be obtained. For $x \in \mathbb{R}$, let

$$f(x) = x + \cos x \sin x \quad \text{and} \quad g(x) = e^{\sin x}(x + \cos x \sin x).$$

1. Show $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$.
2. Show $f'(x) = 2(\cos x)^2$ and $g'(x) = e^{\sin x}\cos x[2\cos x + f(x)]$.
3. Show $\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}$ if $\cos x \neq 0$ and $x > 3$.
4. Show $\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0$ and yet the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does *not* exist.

Solution. As given by Ross, the implicit hypothesis that $g'(x) \neq 0$ for x near s is important.
(1) We know $e^x, \cos x, \sin x, x$ are all continuous on their domains. By continuity theorems, we have:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} x + \left(\lim_{x\to\infty}\cos x\right)\left(\lim_{x\to\infty}\sin x\right)\\ &\geq \lim_{x\to\infty} x - 1\cdot 1\\ &= \boxed{+\infty}\end{aligned}$$

and similarly, because $0 < e^{-1} \leq e^{\sin x} \leq e$,

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left[e^{\sin x} f(x)\right] = \lim_{x\to\infty} e^{\sin x} \cdot \lim_{x\to\infty} f(x) = \boxed{+\infty}$$

(2) Now by derivative theorems, we know $x, \cos x, \sin x$ are infinitely differentiable on $\mathbb{R}$, so:

$$f'(x) = 1 - \sin^2 x + \cos^2 x = \boxed{2\cos^2 x},$$

using the pythagorean trigonometric identity. Alternatively, we could notice $\cos x \sin x = \frac{1}{2}\sin(2x)$, but for our purposes we don't need to use this. Now, we know $e^x, \sin x, f(x)$ are differentiable on $\mathbb{R}$, so we have:

$$\begin{aligned}g'(x) &= \left[e^{\sin x}\right]' f(x) + e^{\sin x} f'(x)\\ &= \cos x e^{\sin x} f(x) + e^{\sin x}[2\cos^2 x]\\ &= \boxed{e^{\sin x}\cos x[2\cos x + f(x)]}.\end{aligned}$$

(3) Now suppose $x > 3$ (so that $2\cos x + f(x) > 0$) and $\cos x \neq 0$. Then from the above in (2) we can cancel $\cos x$ to get:

$$\frac{f'(x)}{g'(x)} = \frac{2\cos x \cdot \cancel{\cos x}}{e^{\sin x}\cancel{\cos x}[2\cos x + f(x)]} = \boxed{\frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}}$$

(4) By continuity of $e^x, \sin x, \cos x$, we have:

$$\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = \frac{\lim_{x\to\infty} 2e^{-\sin x}\cos x}{\lim_{x\to\infty} 2\cos x + \lim_{x\to\infty} f(x)},$$

and notice $0 < \frac{2}{e} \leq 2e^{\sin x}\cos x \leq 2e$, so the numerator is bounded yet $\lim_{x\to\infty} f(x) = +\infty$, so we have:

$$0 = \frac{2/e}{\infty} \leq \lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} \leq \frac{2^e}{\infty} = 0,$$

so we conclude $\boxed{\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0}$. Now consider $x > 3$ in which case the following limit **does not exist**:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\cancel{(x + \cos x \sin x)}}{e^{\sin x}\cancel{(x + \cos x \sin x)}} = e^{\lim_{x\to\infty}(-\sin x)}$$

because for example, taking $a_n := n\pi$ and $b_n := n\pi + \pi/2$ gives $\lim \frac{f(a_n)}{g(a_n)} \neq \lim \frac{f(b_n)}{g(b_n)}$.

□

Problem 31.1. Find the Taylor series for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

Solution. Ross gives in Example 1 (page 252) that "we assume the familiar differentiation properties of $e^x, \sin x$, etc". Hence $\cos x$ falls under this list of familiar properties, whose Taylor series about 0 is

$$\cos x := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \forall_{x \in \mathbb{R}}.$$

If this is not allowed and the expected solution to this is to take the Taylor series of $\sin x$ and differentiate once to get the Taylor series of $\cos x$, we mention that this textbook is bad but do exactly so:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \forall_{x \in \mathbb{R}},$$

as given in Example 1 on Page 252. Taking the derivative across gives the desired Taylor expansion of $\cos x$ about $x = 0$.
Now to see why this Taylor series converges for all $x \in \mathbb{R}$, recall Ross 23.1. Let

$$\beta = \limsup \left| \frac{(-1)^n}{(2n)!} \right|^{1/n} = \limsup \left| \frac{1}{[(2n)!]^{1/n}} \right| = 0$$
$$\implies R := +\infty,$$

so we conclude $\cos x$ has a radius of convergence of $+\infty$ as required. □

Problem 31.4. Consider a, b in $\mathbb{R}$ where $a < b$. Show there exists infinitely differentiable functions f_a, g_b, $h_{a,b}$ and $h^*_{a,b}$ on $\mathbb{R}$ with the following properties.

1. $f_a(x) = 0$ for $x \leq a$ and $f_a(x) > 0$ for $x > a$. *Hint*: Let $f_a(x) = f(x-a)$ where f is the function in Example 3 [page 257 in the book].
2. $g_b(x) = 0$ for $x \geq b$ and $g_b(x) > 0$ for $x < b$.
3. $h_{a,b}(x) > 0$ for $x \in (a,b)$ and $h_{a,b}(x) = 0$ for $x \notin (a,b)$.
4. $h^*_{a,b}(x) = 0$ for $x \leq a$ and $h^*_{a,b}(x) = 1$ for $x \geq b$. *Hint*: Use the function $\frac{f_a}{f_a + g_b}$.

You may assume, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. The same applies to the quotient provided the denominator never vanishes (never zero).

Solution. (1) Taking the hint, let $f(x) := \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0 \end{cases}$ as done in Example 3, page 257 of Ross. Notice that this function is constructed particularly to catch the case when $x \leq 0$, for which $e^{-1/x}$ is not defined.
Then let

$$f_a(x) := f(x-a) = e^{-1/(x-a)},$$

and Example 3 showed this function $f_a(x)$ is infinitely differentiable and satisfies the desired properties.

(2) Take $g_b(x) := f_a(-x)$, and this has the desired property when setting $a \leftarrow b$ (assignment operator). Or perhaps more explicitly, because technically $a < b$, set $g_b(x) := f(-(x-b))$.

(3) Let $h_{a,b}(x) := f_a(x) \cdot g_b(x)$ and notice that if $x \leq a$, then $f_a(x) = 0$ by (1) above and hence $h_{a,b}(x) = 0$. Now, if $x \in (a,b)$, then $f_a(x) > 0$ and $g_b(x) > 0$, so $h_{a,b}(x) > 0$. Lastly, if $b \leq x$, then $g_b(x) = 0$ by (2) above. Because $f_a(x)$ and $g_b(x)$ are infinitely differentiable on $\mathbb{R}$, we conclude (by the allowed assumption above) that $h_{a,b}(x)$ is as well, and we satisfy the desired properties.

(4) Defining $h^*_{a,b}(x) := \frac{f_a(x)}{f_a(x)+g_b(x)}$, we see that if $x \leq a$, we have

$$h^*_{a,b}(x) = \frac{0}{0 + g_b(x)} = 0,$$

because $g_b(x) > 0$ in this case. Now if $x \geq b$, then we have

$$h^*_{a,b}(x) = \frac{f_a(x)}{f_a(x) + 0} = 1,$$

where $f_a(x) > 0$. Further, notice that in this case, $x \geq b > a$, so $x - a > 0$ (this is important), and so $f_a(x) = f(x-a) = e^{-1/(x-a)}$ and is not only infinitely differentiable but infinitely positive. To see this, we formalize via induction. Consider the set $U \in \mathbb{N}$ of orders of derivatives n for which $f_a^{(n)}(x) > 0$, **given the condition** $x > a$. First,

$$f'_a(x) = f'(x-a) = [e^{-1/(x-a)}]' = (x-a)^{-2}e^{-1/(x-a)} > 0,$$

as $x > a \implies (x-a) > 0$ and $e^{-1/(x-a)} > 0$ hence $1 \in U$. Now assume $n \in U$, so that $f_a^{(n)}(x) > 0$. Then

$$f_a^{(n+1)}(x) = (x-a)^{-2} f_a^{(n)}(x) > 0,$$

and hence $n \in U \implies n+1 \in U$, and by Peano's induction axiom $U = N$, so we conclude $x \geq b > a$ implies $h^*_{a,b}(x) = 1$. Our construction has the required properties. □

Problem 32.2. Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

1. Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.
2. Is f integrable on $[0, b]$?

Solution. (1) Take a general subinterval $[t_{k-1}, t_k]$ of some arbitrary but fixed partition $P = \{0 = t_0 < t_1 < \cdots < t_n = b\}$ of $[0, b]$. Notice that

$$m(f, [t_{k-1}, t_k]) = \inf\{f(x) : x \in [t_{k-1}, t_k]\} = 0,$$

by denseness of irrationals in an interval in $\mathbb{R}$. Then for any partition P, we have

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 0 = 0$$
$$\implies L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\} = \sup\{0, \ldots, 0\} = \boxed{0}.$$

On the other hand, for the same (arbitrary) partition P as above, notice $M(f, [t_{k-1}, t_k]) = \sup\{f(x) : x \in [0, b]\} = \sup\{id_x(x) : x \in [0, b]\} = M(id_x, [t_{k-1}, t_k])$, where $id_x(x) = x, \forall x$ denotes the identity operator (function) on $\mathbb{R}$. Because this is true for arbitrary partition and $[t_{k-1}, t_k]$, we conclude $U(f, P) = U(id_x, P)$. However, we proved in the Examples that $id_x(x) = x$ is integrable with $U(id_x) = \frac{1}{2}b^2$. Hence we conclude

$$U(f) := \inf\{U(f, P) : P \text{ of } [0, b]\} = \inf\{U(id_x, P) : P \text{ of } [0, b]\} = \boxed{\frac{1}{2}b^2}.$$

(2) Because $0 = L(f) \neq U(f) = \frac{1}{2}b^2$, no, f is **not integrable** on $[0, b]$.

□

Problem 32.5. Use Exercise 4.8 [page 27 in the book] to prove Theorem 32.4. Specify the sets S and T in this case.

Exercise 4.8. Let S, T be nonempty subsets of $\mathbb{R}$ with the following property:

$$\forall_{s \in S, t \in T}, \quad s \leq t$$

(a) Observe S is bounded above and T is bounded below.
(b) Prove $\sup S \leq \inf T$.
(c) Give an example of such sets S, T with $S \cap T$ nonempty.
(d) Give an example of sets S, T where $\sup S = \inf T$ and $S \cap T = \{\}$.

Definition: Riemann Integral -

Let f be bounded on $[a, b]$. For $S \subset [a, b]$, denote:

$$M(f, S) := \sup\{f(x) : x \in S\}, \qquad m(f, S) := \inf\{f(x) : x \in S\}.$$

Define a **partition** P of $[a, b]$ is any finite ordered subset with the form:

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

Define the **upper Darboux sum** $U(f, P)$ of f with respect to P as the sum

$$U(f, P) := \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \leq M(f, [a, b]) \cdot (b - a)$$

and the **lower Darboux sum** $L(f, P)$ as

$$L(f, P) := \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \geq m(f, [a, b]) \cdot (b - a)$$

so

$$m(f, [a, b]) \cdot (b - a) \leq L(f, P) \leq U(f, P) \leq M(f, [a, b]) \cdot (b - a) \tag{1}$$

and define the **upper Darboux integral** $U(f)$ of f over $[a, b]$ as:

$$U(f) := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and define the **lower Darboux integral** $L(f)$ of f over $[a, b]$ as:

$$L(f) := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

where (1) above gives that $U(f), L(f) \in \mathbb{R}$. If $L(f) = U(f)$, we say f is **integrable** on $[a, b]$.

Ross Theorem 32.4. If f is bounded on $[a, b]$, then $L(f) \leq U(f)$.

Solution. Take $S := L(f, P) \subset \mathbb{R}$ and $T := U(f, P) \subset \mathbb{R}$ for valid partitions P. Then from (1) in the definition above, we have $s \leq t$ for all $s \in S$ and for all $t \in T$. By Exercise 4.8 part b, we have $\sup S \leq \inf T$, which by the definition above, gives precisely:

$$\boxed{L(f) := \sup L(f, P) = \sup S \leq \inf T = \inf U(f, P) =: U(f)}$$

□