Math 104: Homework 10
Solutions

The following problems are taken from the textbook, and listed according to the same numbering:

27.1 Prove Theorem 27.5 from Theorem 27.4. *Hint*: Let $\phi(x) = (b-a)x + a$ so that ϕ maps $[0,1]$ onto $[a,b]$. If f is continuous on $[a,b]$ then $f \circ \phi$ is continuous on $[0,1]$.

Solution: Let f be a continuous function on $[a,b]$. For convenience, set $d = b - a > 0$. As in the hint, define $\phi(x) = dx + a$ (with its domain restricted to $[0,1]$). Then as in the hint it is clear that $\phi([0,1]) = [a,b]$. This means that $f \circ \phi$ is a function on $[0,1]$, and since ϕ is also clearly continuous we see $f \circ \phi$ is continuous on $[0,1]$ by Theorem 17.5. So by Theorem 27.4 we see that $g_n \to f \circ \phi$ uniformly on $[0,1]$, where $g_n = B_n[f \circ \phi]$. Note that ϕ has an inverse function $\phi^{-1} : [a,b] \to [0,1]$ defined by

$$\phi^{-1}(x) = \frac{x-a}{d}$$

which is also linear and hence continuous. Now define $p_n = g_n \circ \phi^{-1}$. Then we indeed have p_n is a function on $[a,b]$. In fact, we claim p_n is a polynomial for each n. To see this, note that for any $x \in [a,b]$ we have

$$\begin{aligned} p_n(x) &= (B_n[f \circ \phi])(\phi^{-1}(x)) \\ &= \sum_{k=0}^{n} (f \circ \phi)\left(\frac{k}{n}\right) \left(\phi^{-1}(x)\right)^k \left(1 - \phi^{-1}(x)\right)^{n-k} \\ &= \sum_{k=0}^{n} f\left(d \cdot \frac{k}{n} + a\right) \left(d^{-1}x - d^{-1}a\right)^k \left(1 - d^{-1}x + d^{-1}a\right)^{n-k}. \end{aligned}$$

In the expression on the final line above, it is clear that $p_n(x)$ is polynomial, as the factors $(d^{-1}x - d^{-1}a)$ and $(1 - d^{-1}x + d^{-1}a)$ are each linear polynomials, so the products in each summand are polynomial and hence p_n itself is as well.

Now, to see that $p_n \to f$ uniformly on $[a,b]$, let $\varepsilon > 0$. Since we already know $g_n \to f \circ \phi$ uniformly on $[0,1]$ we can take N such that for $n > N$ we have $|g_n(x) - f \circ \phi(x)| < \varepsilon$ for $n > N$ and $x \in [0,1]$. We claim that for this same choice of N we have $|p_n(x) - f(x)| < \varepsilon$ for $x \in [a,b]$. To see this, let $n > N$ and $x \in [a,b]$. Then we see that $\phi^{-1}(x) \in [0,1]$, so by assumption we have

$$|g_n(\phi^{-1}(x)) - (f \circ \phi)(\phi^{-1}(x))| < \varepsilon,$$

but then note that by definition we have $g_n(\phi^{-1}(x)) = (g_n \circ \phi^{-1})(x) = p_n(x)$, and also $(f \circ \phi)(\phi^{-1}(x)) = f(\phi(\phi^{-1}(x))) = f(x)$, so indeed

$$|p_n(x) - f(x)| < \varepsilon$$

for $x \in [a,b]$ and $n > N$, as claimed.

27.3 Show there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to f if

(a) $f(x) = \sin x$,

Solution: As a lemma, we will show that if $p(x)$ is any polynomial function on $\mathbb{R}$, then $\lim_{x\to\pm\infty} |p(x)| = \infty$. The argument for $x \to -\infty$ is almost identical to that for $x \to \infty$, so we focus on the latter. Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial (where of course $a_n \neq 0$). Note that we can write

$$|p(x)| = |x|^n \left| a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \ldots + \frac{a_0}{x^n} \right|.$$

Now let (x_k) be a sequence with $x_k \to \infty$. We will show that $|p(x_k)| \to \infty$, which then means $\lim_{x\to\infty} |p(x)| = \infty$ by definition. To see this, note that in the above expression the factor

$$\left| a_n + \frac{a_{n-1}}{x_k} + \frac{a_{n-2}}{x_k^2} + \ldots + \frac{a_0}{x_k^n} \right| \xrightarrow[k\to\infty]{} |a_n|,$$

and we have $0 < |a_n| < \infty$ by assumption. On the other hand clearly we have $|x_k|^n \to \infty$, so by Theorem 12.1 we see that indeed $\lim |p(x_k)| = \infty$, so $\lim_{x\to\infty} |p(x)| = \infty$.

Now, suppose (p_n) is any sequence of polynomials converging to $\sin x$ on $\mathbb{R}$ (in fact, strictly speaking, the assumption of convergence isn't necessary for our purposes here but we'll make it anyway because it feels natural). To see that $p_n \to \sin x$ is not uniform, consider $\varepsilon = 1$. Then for any n, we have by the preceding argument that there is some x for which $|p_n(x)| > 2$. But then this means

$$|p_n(x) - \sin x| \geq |p_n(x)| - |\sin x| > 2 - 1 = \varepsilon,$$

since $|p_n(x)| > 2$ and $|\sin x| \leq 1$. So, for this ε, no N can exist for which $|p_n(x) - \sin x| < \varepsilon$ for all $n > N$ and all $x \in \mathbb{R}$, showing that indeed $p_n(x)$ does not converge uniformly to $\sin x$.

(b) $f(x) = e^x$.

Solution: Note that for $x \leq 0$ we have $|f(x)| < 1$. So in similar fashion to the previous part, let (p_n) be any sequence of polynomials converging to e^x on $\mathbb{R}$. Then consider $\varepsilon = 1$. Since for any polynomial p we have by the previous part that $\lim_{x\to-\infty} |p(x)| = \infty$, for any n we can find $x < 0$ such that $|p_n(x)| > 2$. But then by an argument parallel to that in the previous part, we see that $|p_n(x) - f(x)| > \varepsilon$ for this choice of x, showing that the convergence cannot be uniform.

27.6 The Bernstein polynomials were defined for any function f on $[0, 1]$. Show that if $B_n f \to f$ uniformly on $[0, 1]$, then f is continuous on $[0, 1]$.

Solution: Note that since the Bernstein polynomials are polynomials, they are automatically continuous on $[0, 1]$ (if you like, this follows inductively by finitely many applications of the sum and product rules for continuous functions, starting from the base cases that $f(x) = x$ and constant functions are continuous). So if $B_n f \to f$ uniformly on $[0, 1]$ we must have that f is continuous by Theorem 24.3.

28.4 Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 28.3 and 28.4 to show f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and $\cos x$ is its derivative.

Solution: Note that since 1 and x are differentiable functions on $\mathbb{R}$ and if $a \neq 0$ then $x \neq 0$ at $x = a$ we have that $g(x) = 1/x$ is differentiable at a by Theorem 28.3(iv). Using the formula, we see that $g'(a) = -1/a^2$. We also know sine is differentiable everywhere, so by Theorem 28.4 we have that $\sin \circ g$ is differentiable at a, and

$$(\sin \circ g)'(a) = \cos(g(a))g'(a) = -\cos(\frac{1}{a})/a^2.$$

Lastly, since x is differentiable everywhere, by two applications of (iii) we have that f is differentiable for $x \neq 0$ and the value at $x = a$ is

$$f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}.$$

(b) Use the definition to show f is differentiable at $x = 0$ and $f'(0) = 0$.

Solution: To show f is differentiable at $x = 0$ and $f'(0) = 0$, by definition, we must show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0,$$

or equivalently (since $f(0) = 0$)

$$\lim_{x \to 0} \frac{f(x)}{x} = 0.$$

Now note that for $x \neq 0$ we have $f(x)/x = x \sin(1/x)$, and since the existence and value of the above limit depend only on the values of $f(x)/x$ away from $x = 0$, we have reduced the problem to showing

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

To see this, let $\varepsilon > 0$. Then take $\delta < \varepsilon$. Then for $0 < |x| < \delta$ we clearly have

$$\left| x \sin\left(\frac{1}{x}\right) \right| \leq |x| < \delta < \varepsilon,$$

so indeed we have $\lim_{x \to 0} f(x)/x = 0$, meaning $f'(0)$ exists and $f'(0) = 0$.

(c) Show f' is not continuous at $x = 0$. [Max's note: you should really study this fact closely. I think it should defy almost anyone's intuition that f' can exist at a point and yet fail to be continuous there.]

Solution: Consider the sequence $x_n = 1/(\pi n)$, which has $\lim x_n = 0$. Using the formula from part (a), we see that

$$f'(x_n) = -\cos(\pi n) = (-1)^{n+1},$$

so $\lim f'(x_n)$ does not exist, showing f' is not continuous at 0.

28.5 Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$, and $g(x) = x$ for $x \in \mathbb{R}$.

(a) Observe f and g are differentiable on $\mathbb{R}$.

Solution: We showed in the previous problem that f is differentiable everywhere, and g is trivially differentiable.

(b) Calculate $f(x)$ for $x = \frac{1}{\pi n}$, $n = \pm 1, \pm 2, \ldots$.

Solution: We have

$$f\left(\frac{1}{\pi n}\right) = \frac{1}{\pi^2 n^2} \sin(\pi n) = 0.$$

(c) Explain why $\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$ is meaningless.

Solution: The expression is meaningless because the denominator $f(x) - f(0)$ (equal to $f(x)$ because $f(0) = 0$) is zero at all the points $x = 1/(n\pi)$ where $n \in \mathbb{Z} \setminus \{0\}$, so there is no open interval $(-\varepsilon, \varepsilon)$ containing zero on which the expression in the limit is defined (i.e. we always have $1/(n\pi) \in (-\varepsilon, \varepsilon)$ for sufficiently large n).

28.8 Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) Prove f is continuous at $x = 0$.

Solution: Let $\varepsilon > 0$. Take $\delta < \sqrt{\varepsilon}$. Now let $|x - 0| = |x| < \delta$. We claim $|f(x) - f(0)| = |f(x)| < \varepsilon$. To see this, note that for any x with $|x| < \delta$ either $f(x) = x^2$ or $f(x) = 0$, and in both cases we have $|f(x)| \leq |x|^2 < \delta^2 < \varepsilon$, so indeed $|f(x)| < \varepsilon$ as claimed. This shows f is continuous at 0.

(b) Prove f is discontinuous at all $x \neq 0$.

Solution: Let $x \neq 0$. There are two cases: either x is rational or it is irrational. If x is rational, then we have $f(x) = x^2 \neq 0$. However, in this case we can consider a sequence of points (x_n) which are all irrational such that $x_n \to x$ (this is easily constructed since the irrational numbers are dense). For this sequence, we have $f(x_n) \to 0 \neq f(x)$, so f cannot be continuous at x. On the other hand, if x is irrational, $f(x) = 0$, but we can similarly take a sequence of points (x_n) which are all rational with $x_n \to x$, and then $f(x_n) \to x^2 \neq 0 = f(x)$, so again f is not continuous at $x = 0$.

(c) Prove f is differentiable at $x = 0$. *Warning*: You cannot simply claim $f'(x) = 2x$. [Max's note: again, you should take special notice of the pathology exhibited in this problem. Recall part 1 of the T/F section on midterm 2 – the conclusion was that a function can be continuous at a single point while failing to be continuous at any other points nearby (or indeed anywhere else at all). In this case we something even stronger – hence even less intuitive – that f can be not only continuous but *differentiable* at a point, while failing EVEN to be continuous at a single other point. In fact, this example can be generalized quite broadly to show that a function can have any finite number of derivatives at a point while being continuous nowhere else (with only slightly more difficulty, one can even conjure an example of a function that is smooth – having infinitely many derivatives – at a single point while being continuous nowhere else). This is a powerful testament to the local nature of the properties of continuity and differentiability. CORRECTION 8/14: This is wrong. The way we have defined the derivative, if $f^{(n)}$ exists at a point, then in fact all lower derivatives must exist in a neighborhood of that point, meaning continuity certainly holds in a neighborhood if $n > 1$. See the posted announcement.]

Solution: We seek to show $\lim_{x\to 0} f(x)/x = 0$, which suffices to show that f is differentiable at 0 and moreover $f'(0) = 0$. Note that when $x \neq 0$ we have $f(x)/x = x$ when x is rational and $f(x)/x = 0$ when x is irrational, and in either case we have $|f(x)/x| \leq |x|$. So let $\varepsilon > 0$ and take $\delta < \varepsilon$. Then for $0 < |x| < \delta$ we have $|f(x)/x| \leq |x| < \delta < \varepsilon$, thus verifying definitionally that $\lim_{x\to 0} f(x)/x = 0$, and therefore that f' exists at 0 and $f'(0) = 0$.

29.5 Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

Solution: First we claim that f is differentiable at all $a \in \mathbb{R}$, and $f'(a) = 0$. To see this, let $a \in \mathbb{R}$ and we will show

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = 0.$$

To see this, let $\varepsilon > 0$ and take $\delta < \varepsilon$. Then for $0 < |x - a| < \delta$ we have by the assumption in the problem that

$$|f(x) - f(a)| \leq |x - a|^2,$$

so after dividing both sides by $|x-a|$ we see that

$$\left|\frac{f(x)-f(a)}{x-a}\right| \leq |x-a| < \delta < \varepsilon,$$

so indeed f is differentiable at a and $f'(a) = 0$. Now since $f'(x) = 0$ for all x, we have by Theorem 29.4 that f is constant on any open interval (a, b) with a, b finite. But this directly implies that f is constant on $\mathbb{R}$, since if $f(x_1) \neq f(x_2)$ for some $x_1 < x_2$ then we would have that f is not constant on the open interval $(x_1 - 1, x_2 + 1)$, contradicting what we just proved.

29.7 (a) Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b. [Max's note: Your solution should make NO REFERENCE to the notion of an integral whatsoever. Consider trying to show that $f'(x) = a$ for some $a \in \mathbb{R}$.]

Solution: Note that the argument at the end of exercise 29.5 above shows that if a function f is differentiable on any open interval I (in particular where one or both of the "endpoints" of I may be infinite) with $f' = 0$ on all of I then we still have that f is constant. So, in this case, since $(f')' = 0$ on I we must have that f' is constant, say $f'(x) = a$ for some $a \in \mathbb{R}$. Now consider the function $g(x) = ax$ on I. We straightforwardly have that $g'(x) = a$ for $x \in \mathbb{I}$, so that $g' = f'$ on I. Then by Corollary 29.5 (extended to possibly unbounded intervals by the same technique as above) we have that there is a constant, call it b in this case, so that $f(x) = g(x) + b$ for $x \in I$. But this means $f(x) = ax + b$ for $x \in I$, as claimed.

(b) Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim.

Solution: We claim $f(x) = ax^2 + bx + c$ for some constants $a, b, c \in \mathbb{R}$. To see this, first note that since $(f')'' = 0$ we have by the previous part that $f'(x) = 2ax + b$ for some $a, b \in \mathbb{R}$. Now consider the function $g(x) = ax^2 + bx$. We see that $g'(x) = 2ax + b$ on I, so we have $g' = f'$ on I meaning there is some constant, call it c, so that $f(x) = g(x) + c$ for $x \in I$. This means $f(x) = ax^2 + bx + c$, as desired.