Math 104, Summer 2019

PSET #10 (due Thursday 8/8/2019)

Ross 27.4 For every continuous function f on $[0,1]$, we have $B_n f \to f$ uniformly on $[0,1]$.

Ross 27.5 Weierstrass' Approximation Theorem. Every continuous function on a closed interval $[a,b]$ can be uniformly approximated by polynomials on $[a,b]$.
In other words, given a continuous function f on $[a,b]$, there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a,b]$.

Problem 27.1. Prove Theorem 27.5 from Theorem 27.4.

Solution. As given by the hint, let $\phi(x) := (b-a)x + a$, so that $\phi : [0,1] \to [a,b]$. If f is continuous on $[a,b]$, then $f \circ \phi$ is continuous on $[0,1]$ (by the theorem that states that the composition of continuous functions is continuous). If this is true, then $f \circ \phi$ is continuous on the bounded interval $[0,1]$, it is uniformly continuous. Then Ross 27.4 gives us the existence of one such uniform approximation to $f \circ \phi$ on $[a,b]$, namely the Bernstein polynomials.

Precisely, there exists a sequence $(B_n f)$ of polynomials such that $B_n f \to f \circ \phi$ uniformly on $[0,1]$. Knowledge that $\phi(x)$ is a bijection and thus has a unique inverse is unnecessary–it suffices to symbolically show that $\theta(x) = \frac{x-a}{b-a}$ satisfies:

$$\theta \circ \phi(x) = \frac{[(b-a)x + a] - a}{b-a} = \phi \circ \theta(x) = (b-a)\left[\frac{x-a}{b-a}\right] + a = x,$$

provided $a \neq b$ (which is given as $[a,b]$ is an interval, assumed non-degenerate and hence not a point). Then $B_n f \circ \theta$ is a composite polynomial of polynomials and hence itself a polynomial, with $B_n f \circ \theta \to f \circ \phi \circ \theta = f$ uniformly on $[a,b]$, as desired. $\square$

Problem 27.3. Show there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to f if
1. $f(x) = \sin x$,
2. $f(x) = e^x$.

Solution. (1)First recall that $-1 \leq |\sin x| \leq 1$, so that we can (ab)use this fact. We proved before that non-constant polynomials have left and right limits $\pm\infty$. Precisely, we proved before that a polynomial is bounded on $\mathbb{R}$ if and only if it is constant. Surely, if a sequence of polynomials $(p_n) \to \sin x$ uniformly on $\mathbb{R}$, then the sequence of polynomials converges to a bounded function. However, this is only the case if $(p_n) \to c$, for some constant polynomial c. Now, $\sin x \neq c$, so we conclude there cannot exist a sequence of polynomials converging uniformly to $\sin x$.

(2) If there is a sequence (p_n) of polynomials converging uniformly to e^x on $\mathbb{R}$, then $\forall_{x \in \mathbb{R}}$, for any $\epsilon > 0$, we have for all $n > N$ that

$$|e^x - p_n| < \epsilon.$$

Recall that a polynomial by definition has a finite degree, and hence we can write:

$$p_n = c_0 + c_1 x^1 + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

for the degree-n polynomial p_n, and for some $c_i \in \mathbb{R}$. Take $\epsilon := 1$ so that assumed uniform convergence of $(p_n) \to e^x$ gives:

$$\left| e^x - \sum_{j=0}^{n-1} c_j x^j \right| < 1, \forall x \in \mathbb{R}.$$

Now, we know that one definition of e^x for $x \geq 0$ is

$$e^x := \sum_{j=0}^{\infty} \frac{x^j}{j!},$$

so we write that for the same n,

$$\begin{aligned} \frac{1}{n!}x^n &\leq \left| e^x + \sum_{j=0}^{n-1} c_j x^j - \sum_{j=0}^{n-1} c_j x^j \right| \\ &\leq \left| e^x - \sum_{j=0}^{n-1} c_j x^j \right| + \left| \sum_{j=0}^{n-1} |c_j| x^j \right| \\ \implies \forall x > 0: \qquad \frac{1}{n!}x^n - \sum_{j=0}^{n-1} |c_j| x^j &\leq \left| e^x - \sum_{j=0}^{n-1} c_j x^j \right| < 1, \end{aligned}$$

where

$$\sum_{j=0}^{n-1} |c_j| x^j \geq 0,$$

so

$$\frac{1}{n!}x^n < 1, \forall_{x \in \mathbb{R}}.$$

Take $x = n$, and this is false, so we have a contradiction, and it must be the case that there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to e^x. □

Problem 27.6. The Bernstein polynomials were defined for any function f on $[0, 1]$. Show that if $B_n f \to f$ uniformly on $[0, 1]$, then f is continuous on $[0, 1]$.

Solution. Ross 27.4 gives that for every continuous function f on $[0, 1]$, we have $B_n f \to f$ uniformly on $[0, 1]$. Now we wish to prove the converse, in that if $B_n f \to f$ uniformly on $[0, 1]$, then f is continuous on $[0, 1]$.

This result is trivial. Notice that $B_n f$ is a sequence of polynomials, and hence every polynomial in its sequence is a polynomial (which we have proved is continuous on $\mathbb{R}$). Now because $B_n f$ converges to f, the limit function $\lim B_n f$ equals f, and hence $f = \lim B_n f$ is continuous. □

Problem 28.4. Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

1. Use Theorems 28.3 and 28.4 to show f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and $\cos x$ is its derivative.
2. Use the definition to show f is differentiable at $x = 0$ and $f'(0) = 0$.
3. Show f' is not continuous at $x = 0$. [Max's note: you should really study this fact closely. I think it should defy almost anyone's intuition that f' can exist at a point and yet fail to be continuous there.]

Solution. (1) We have $f(x) := \begin{cases} 0, & x = 0 \\ x^2 \sin\left(\frac{1}{x}\right), & x \neq 0. \end{cases}$

We are given that $\sin x$ is differentiable and $\cos x$ is its derivative. Take $x \neq 0$ and the derivative theorems give us:

$$\begin{aligned} f'(x) &= 2x \sin\left(\frac{1}{x}\right) + x^2 \frac{d}{dx} \sin\left(\frac{1}{x}\right) \\ &= 2x \sin\left(\frac{1}{x}\right) + \cancel{x^2} \cos\left(\frac{1}{x}\right)(-\cancel{x^{-2}}) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \end{aligned}$$

so because $a \neq 0$, we have $\boxed{f'(a) = 2a \sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right)}$.

(2) The definition, if the limit makes sense, gives:

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \\ &= \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \end{aligned}$$

where we have our second line from the fact that our limit does not take subsequences exactly at $x = 0$ but rather clusters of points near it. Invoking the fact that $\sin t$ is Lipschitz and hence bounded (we proved this in a prior homework), this gives

$$f'(0) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x = 0,$$

so we conclude $\boxed{f'(0) = 0}$, which was to be shown.

(3) Now we take the same sequence x_n as in the next problem with $\frac{1}{x_n} = \{\pi, 2\pi, 3\pi, \ldots, n\pi, \ldots\}$. Using our expression from (1) above, notice that

$$f'(x_n) = 2x_n \sin\left(\frac{1}{x_n}\right) - \cos\left(\frac{1}{x_n}\right)$$

gives a sequence $f'(x_n) = \{1, -1, 1, -1, \ldots\}$, where the oscillation comes from $-\cos\left(\frac{1}{x_n}\right)$ and the fact that for all n, $\sin x_n = 0$. So $\lim_{n \to \infty} f'(x_n)$ does not exist (for this sequence) and because $x_n \to 0$, $f'(x)$ is **not continuous** at $x = 0$. □

Problem 28.5. Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$, and $g(x) = x$ for $x \in \mathbb{R}$.

1. Observe f and g are differentiable on $\mathbb{R}$.
2. Calculate $f(x)$ for $x = \frac{1}{\pi n}$, $n = \pm 1, \pm 2, \ldots$.
3. Explain why $\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$ is meaningless.

Solution. We are given:

$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

$$g(x) := x, \quad \forall x \in \mathbb{R}.$$

(1) As requested, we observe f, g are differentiable on $\mathbb{R}$ and meditate on this fact.

(2) Taking $x_n = \frac{1}{\pi n}$ gives/ the sequence $\frac{1}{x_n} = \{\pi, 2\pi, 3\pi, \ldots\}$ and hence the desired sequence of function values is:

$$\boxed{f(x_n) = \{0, 0, 0, \ldots\}}.$$

(3) Now take

$$h(x) := \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$$

and recall that $\lim_{x\to 0} h(x)$ exists if the function h is defined on an interval enclosing 0. However, some tail end of the sequence x_n as defined in (2) exists in any such interval as $x_n \to 0$, so the limit $\lim_{x\to 0} h(x_n)$ is undefined and hence the limit cannot exist (because all subsequences of a convergent sequence must converge to the same limit).

Perhaps more rigorously a la Analysis, taking any interval $(-\xi, \xi)$ with $\xi > 0$ contains infinitely many indices n for which $x_n \in (-\xi, \xi)$, and hence $f(x_n) = 0$. So such an interval does not exist where h as written above is actually defined. □

Problem 28.8. Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.
1. Prove f is continuous at $x = 0$.
2. Prove f is discontinuous at all $x \neq 0$.
3. Prove f is differentiable at $x = 0$.

Warning: You cannot simply claim $f'(x) = 2x$. [Max's note: again, you should take special notice of the pathology exhibited in this problem. Recall part 1 of the T/F section on midterm 2 – the conclusion was that a function can be continuous at a single point while failing to be continuous at any other points nearby (or indeed anywhere else at all). In this case we something even stronger – hence even less intuitive – that f can be not only continuous but *differentiable* at a point, while failing EVEN to be continuous at a single other point. In fact, this example can be generalized quite broadly to show that a function can have any finite number of derivatives at a point while being continuous nowhere else (with only slightly more difficulty, one can even conjure an example of a function that is smooth – having infinitely many derivatives – at a single point while being continuous nowhere else). This is a powerful testament to the local nature of the properties of continuity and differentiability.]

Solution. (1) To show that $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ x^2, & x \in Q \end{cases}$ is continuous at $x = 0$, fix $\epsilon > 0$ and take $\delta = \sqrt{\epsilon}$. Then for $x \in \mathbb{R}$, $|x - 0| < \delta$ implies:
(i) if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|f(x) - f(0)| = |0 - 0| = 0 < \epsilon$, as required, or
(ii) if $x \in \mathbb{Q}$, then $|f(x) - f(0)| = |x^2 - 0^2| = |x - 0|^2 < \delta^2 = \epsilon$, as required.
We conclude that f as defined above is continuous at 0.

(2) Now we show f is discontinuous at every other point, namely $x \neq 0$. We split into cases.
First consider $x \in \mathbb{R} \setminus \mathbb{Q}$. By density of rationals in $\mathbb{R}$, every open interval must contain a rational number. To disprove continuity at $x \neq 0$, take $\epsilon = x^2 > 0$ and $\delta > 0$. Take $\xi \in \mathbb{Q}$ with $\xi \in (x - \delta, x + \delta)$, so that $f(\xi) = \xi^2 > x^2$ (such choices exist by the above). Then we can have $|x - \xi| < \delta$ yet $|f(x) - f(\xi)| = |0 - \xi^2| = |0 - \xi|^2 > x^2 = \epsilon$, and hence f cannot be continuous at $x \in \mathbb{R} \setminus \mathbb{Q}$.

Now we consider $x \in \mathbb{Q}$. We proved on a previous homework that the irrationals are dense in $\mathbb{R}$, precisely that if $r_1 < r_2$ with $r_1, r_2 \in \mathbb{R}$, there exists some $\xi \in \mathbb{R} \setminus \mathbb{Q}$ with $r_1 < \xi < r_2$. Hence every open interval must contain an irrational number. To disprove continuity at $x \neq 0$, take $\epsilon := \frac{x^2}{2} > 0$ and $\delta > 0$. Take $\xi \in \mathbb{R} \setminus \mathbb{Q}$ with $\xi \in (-\delta, \delta)$. Then we can have $|x - \xi| < \delta$ yet $|f(x) - f(\xi)| = |x^2 - 0| = x^2 > \epsilon = \frac{x^2}{2}$. Hence f is not continuous at nonzero $x \in \mathbb{Q}$.

(3) Now to show f is differentiable at $x = 0$, we claim $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. To see this, fix $\epsilon > 0$ and take $\delta := \epsilon$. If $|x - 0| < \delta$, we again have two cases.
(i) If $x \in \mathbb{Q}$, this gives: $\left|\frac{f(x)}{x-0} - 0\right| = \left|\frac{x^2}{x}\right| = |x| < \delta = \epsilon$ by the above.
(ii) If $x \in \mathbb{R} \setminus \mathbb{Q}$, then this gives: $\left|\frac{f(x)}{x-0} - 0\right| = \left|\frac{0}{x} - 0\right| = 0 < \epsilon$.

We conclude that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ as desired, and so f is differentiable at $x = 0$. In fact, we write $\boxed{f'(0) = 0}$.

□

Ross 29.3 Mean Value Theorem Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists (at least one) $x \in (a, b)$ with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Ross 29.4 Corollary Let f be a differentiable function on (a, b) with $f'(x) = 0, \forall_{x\in(a,b)}$. Then f is a constant function on (a, b).

Problem 29.5. Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

Solution. Fix $a \in \mathbb{R}$ and suppose $x \neq y$ (where if $x = y$), so that the given expression equivalently gives:

$$0 \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|.$$

Hence going from right to left, $x \in B_\epsilon(a)$ (or equivalently $|x - a| < \epsilon$) implies $\frac{f(x)-f(y)}{x-y} \in B_\epsilon(0)$ (or equivalently, $\left|\frac{f(x)-f(y)}{x-y} - 0\right| < \epsilon$), where $B_r(\xi)$ denotes the neighborhood or open ball around ξ with radius r. Then the derivative

$$f'(y) = \lim_{x\to y} \frac{f(x) - f(y)}{x - y} = 0$$

is identically zero everywhere. Hence because $f'(x) = 0$ for all $x \in \mathbb{R}$, Ross 29.4 above gives that f is a constant function on $\mathbb{R}$.

□

Problem 29.7.

1. Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.
2. Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim.

[Max's note: Your solution for (1) should make NO REFERENCE to the notion of an integral whatsoever. Consider trying to show that $f'(x) = a$ for some $a \in \mathbb{R}$.]

Solution. (1) Because $f''(x) = 0, \forall_{x \in I}$, we have that $f'(x)$ is a constant function. Now we set $f'(x) = c$ for some constant $c \in \mathbb{R}$, and define $g(x) := f(x) - cx$, which inherits the twice-differentiability of f as the monomial $-ax$ is infinitely differentiable. Notice then derivative theorems give $g'(x) = f'(x) - c \cdot 1 = c - c = 0$, for all $x \in I$. Because $g'(x)$ is identically zero at all $x \in I$, then g is a constant function, say $g(x) = k$ for some $k \in \mathbb{R}$. Hence:

$$g(x) = k = f(x) - cx$$
$$f(x) = cx + k,$$

which is equivalent to $f(x) = ax + b$ for $a, b \in \mathbb{R}$.
(2) Now we have f is thrice-differentiable on I, and $f'''(x) = 0$ for all $x \in I$. We claim f has the form:

$$f(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$. To see this explicitly, we have $f''' = (f')'' = 0$ gives that $f'(x) = ax + b$ for some $a, b \in \mathbb{R}$. Define $g(x) = f(x) - \left(\frac{a}{2}x^2 + bx\right)$, so that

$$g'(x) = \frac{d}{dx}\left[f(x) - \frac{a}{2}x^2 - bx\right] = f'(x) - (ax + b) = 0$$

Because $g'(x) = 0$ exactly everywhere on I, $g(x)$ is a constant function, so there exists some $c \in \mathbb{R}$ such that $g(x) = c$. Hence

$$g(x) = c = f(x) - \left(\frac{a}{2}x^2 + bx\right)$$
$$f(x) = \frac{a}{2}x^2 + bx + c,$$

as desired. $\square$