Math 104, Summer 2019

PSET #1 (due Thu 6/27/2019)

Problem 1.1. Prove for all positive integers n:

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution. Consider the subset $U \subset \mathbb{N}$ of values n for which this statement is true. We see that

$$1^2 = \frac{1}{6}(1)(1+1)[2(1)+1] = \frac{1 \cdot 2 \cdot 3}{6} = 1 \implies 1 \in U$$
$$1^2 + 2^2 = \frac{1}{6}(2)[2+1][2(2)+1] = \frac{2 \cdot 3 \cdot 5}{6} = 5 \implies 2 \in U$$

Now assume $k \in U$, so that we have $1^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$. Adding $(k+1)^2$ to both sides of this inductive hypothesis, we get:

$$\begin{aligned}1^2 + \cdots + k^2 + (k+1)^2 &= \frac{1}{6}k(k+1)[2k+1] + (k+1)^2 \\ &= (k+1)[\frac{1}{6}k(2k+1) + (k+1)] \\ &= (k+1)[\frac{[2k^2+k] + 6(k+1)}{6}] \\ &= (k+1)[\frac{[2k^2+7k+6]}{6}] \\ &= \frac{1}{6}[k+1][k+2][2k+3] \\ &= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]\end{aligned}$$

We have explicitly shown $k \in U$ implies $k+1 \in U$, so by induction, we have $U = \mathbb{N}$ as desired. □

Problem 1.12. For $n \in \mathbb{N}$, let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$ (and let $0! := 1$). Define

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \text{ for } k = 0, 1, \ldots, n$$

The binomial theorem asserts that

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

(a) Verify the binomial theorem for $n = 1, 2, 3$.
(b) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.
(c) Prove the binomial theorem using mathematical induction and part (b).

Solution. (a) First we simply want to verify that $n = 1, 2, 3$ satisfy the binomial theorem.
($n = 1$) $(a+b)^1 = a^1 + b^1$, which is true.
($n = 2$) $(a+b)^2 = a^2 + 2ab + b^2$, which is true.
($n = 3$) $(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which is true.

(b) Next we're asked to show for $k = 1, 2, \ldots, n$, that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Intuitively, this gives the identity behind Pascal's Triangle. But we'll pretend we don't know that and prove it symbolically.

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} \\
&= \frac{(n!)(n-k+1)}{(n-k+1)!k!} + \frac{(n!)(k)}{(n-k+1)!k!} \\
&= \frac{(n!)[k+n-k+1]}{(n-k+1)!k!} = \frac{(n!)[n+1]}{[(n+1)-k]k!} = \binom{n+1}{k},
\end{aligned}$$

as desired. Because we have a well defined definition for factorial and $\binom{n}{k}$, we have shown our identity for all $k = 1, 2, \ldots, n$.

(c) Now we prove the actual binomial theorem, for all $n \in \mathbb{N}$ (and all $k = 1, \ldots, n$). Consider the set $U \subset \mathbb{N}$ for which the binomial theorem holds. We have already shown $1, 2, 3 \in U$ in part (a). Assume $m \in U$ so that we have

$$(a+b)^m = \sum_{i=0}^{m} \binom{m}{i} a^{m-i} b^i.$$

Consider:

$$\begin{aligned}
(a+b)^{m+1} &= (a+b)(a+b)^m \quad \text{(by definition of iterated power)} \\
&= (a+b)\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^i \quad \text{(because } m \in U\text{)} \\
&= \left[a\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^i\right] + \left[b\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^i\right] \quad \text{(distributivity of mult. over add.)} \\
&= \left[\sum_{i=0}^{m}\binom{m}{i}a^{m+1-i}b^i\right] + \left[\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^{i+1}\right] \quad (\text{`` ''}) \\
&= \left[\binom{m}{0}a^{m+1} + \sum_{i=1}^{m}\binom{m}{i}a^{m+1-i}b^i\right] + \left[\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^{i+1}\right] \quad \text{(pull out a term)} \\
&= \left[\binom{m}{0}a^{m+1} + \sum_{i=0}^{m-1}\binom{m}{i+1}a^{m-i}b^{i+1}\right] + \left[\binom{m}{m}b^{m+1} + \sum_{i=0}^{m-1}\binom{m}{i}a^{m-i}b^{i+1}\right] \quad \text{(adjust left sum index, `` '')} \\
&= \left[\binom{m}{0}a^{m+1} + \binom{m}{m}b^{m+1}\right] + \sum_{i=0}^{m-1}\left[\binom{m}{i+1} + \binom{m}{i}\right]a^{m-1}b^{i+1} \quad \text{(combine terms in summation)} \\
&= \left[\binom{m}{0}a^{m+1} + \binom{m}{m}b^{m+1}\right] + \sum_{i=0}^{m-1}\left[\binom{m+1}{i+1}\right]a^{m-1}b^{i+1} \quad \text{(using Pascal's identity from (b))} \\
&= \left[1a^{m+1} + 1b^{m+1}\right] + \sum_{i=1}^{m}\left[\binom{m+1}{i}\right]a^{m-1}b^{i} \quad \text{(adjust sum index)} \\
&= \sum_{i=0}^{m+1}\binom{m+1}{i}a^{(m+1)-i}b^i,
\end{aligned}$$

which is precisely the binomial theorem for $m+1$. This shows $m \in U \implies (m+1) \in U$, so by induction we have $U = \mathbb{N}$ as desired. $\square$

Problem 2.5. Show $\left[3+\sqrt{2}\right]^{2/3}$ is not a rational number.

Solution. Let $x := \left[3+\sqrt{2}\right]^{2/3}$. We will show x is an algebraic number but not rational. Suppose for contradiction that $x \in \mathbb{Q}$, so then we can write for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and $\gcd\{a, b\} = 1$, $x = \frac{a}{b}$. Then we have:

$$\begin{aligned} x = \frac{a}{b} &= \left[3+\sqrt{2}\right]^{2/3} \\ \left(\frac{a}{b}\right)^3 &= \left[3+\sqrt{2}\right]^2 = [9+2+6\sqrt{2}] \\ a^3 &= [11+6\sqrt{2}]b^3 = 11b^3 + (\sqrt{2})6b^3 \\ a^3 - 11b^3 &= (\sqrt{2})6b^3 \end{aligned}$$

However, we've already proven $\sqrt{2}$ is irrational. Assuming we know a rational number times an irrational number is irrational (lemma 1), it's easy to see the LHS $a^3 - 11b^3 \in \mathbb{Z}$, whereas for $b \in \mathbb{Z}$, the right side cannot be rational.

However, if we have to somehow use the rational zeros theorem (RZT), then we can proceed:

$$\begin{aligned} \left[a^3 - 11b^3\right]^2 &= 2\left[6b^3\right]^2 \quad \text{(squaring both sides)} \\ a^6 + 121b^6 - 22a^3b^3 &= 72b^6 \\ \frac{1}{b^6}[a^6 - 22a^3b^3 + 49b^6] &= 0 \quad \text{(assuming } b \neq 0\text{)} \\ x^6 - 22x^3 + 49 &= 0 \quad \text{(recall that } x := a/b\text{)} \end{aligned}$$

So a is a root to the above monic polynomial in x, which by the RZT gives that if a is rational, it must be one of the following: $\pm1, \pm7, \pm49$, given by the prime factorization of 49. Clearly (by testing), ±1 is not a root. To see -7 is not a root, it's easy to see $(-7)^2 - 22(-7)^3 + 49$ is a sum of all positive terms and thus strictly positive (nonzero). Similarly, if (-49) is a root, then we would have $(-49)^6 - 22(-49)^3 + 49 = 49^6 + 22(49)^3 + 49$, which is again the sum of positive terms (and thus nonzero). The result is similar with $(7)^6 + 49 > 22(7)^3$, so 7 cannot be a root. Now for ±49, consider that if 49 were a root, we can factor the LHS: $49[49^5 - 22(49)^2 + 1]$, where the right bracketed expression is surely positive (nonzero). Because all possible candidates (given by RZT) are not roots, then x which we defined as equal to our expression of interest, $\left[3+\sqrt{2}\right]^{2/3}$, cannot be rational, and we are done. □

Lemma 0.1. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $q \in \mathbb{Q}$. Then the product $\alpha q \notin \mathbb{Q}$ (is not rational).

Proof. Assumed to be known, since this lemma is only for artistic purposes as the above solution is complete with RZT (although the colorful box will get butchered by my terrible printer). By the way this is a lot of homework for PSETS of our frequency, but I understand practice is important. □

Problem 2.7. Show the following irrational-looking expressions are actually rational numbers:
(a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$
(b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

Solution. We'll prove something even stronger. For each expression, we'll not only show that it is rational but also give its value.

(b) Let $a := \sqrt{4+2\sqrt{3}}-\sqrt{3}$. Then we have:

$$\begin{aligned} a+\sqrt{3} &= \sqrt{4+2\sqrt{3}} \\ (a+\sqrt{3})^2 &= 4+2\sqrt{3} \\ a^2+2\sqrt{3}a-2\sqrt{3}-1 &= 0 \end{aligned}$$

Immediately by inspection (or moving 1 over to the right and factoring LHS), we see that $a = 1$ satisfies this equation, so we have $\sqrt{4+2\sqrt{3}}-\sqrt{3} = 1$, which is rational.

(b) Let $b := \sqrt{6+4\sqrt{2}}-\sqrt{2}$. Then we have:

$$\begin{aligned} b+\sqrt{2} &= \sqrt{6+4\sqrt{2}} \\ b^2+2\sqrt{2}b+2 &= 6+4\sqrt{2} \\ b^2+2\sqrt{2}b-4\sqrt{2}-4 &= 0 \end{aligned}$$

By inspection (or moving 4 to RHS and factoring LHS), we see that $b = 2$ satisfies this equation, so we have $\sqrt{6+4\sqrt{2}}-\sqrt{2} = 2$, which is rational. □

Problem 3.3. Prove (iv) and (v) of Theorem 3.1 (textbook page 15).

Theorem 3.1, Ross p.15 (Consequences of Field Properties)

(i) $a + c = b + c \implies a = b$;
(ii) $a \cdot 0 = 0, \quad \forall a$;
(iii) $(-a)b = -ab \quad \forall a, b$;
(iv) $(-a)(-b) = ab \quad \forall a, b$;
(v) $ac = bc, c \neq 0 \implies a = b$;
(vi) $ab = 0 \implies [(a = 0) \cup (b = 0)], \quad \forall a, b, c \in \mathbb{R}$.

Solution. The proofs of (i), (ii), (iii), (vi) are given on page 16. These will be included along with my own proofs of (iv) and (v) for completeness.

(i) $a + c = b + c \implies a = b$;
Proof. This is the right cancellation law. $a+c = b+c \implies (a+c)+(-c) = (b+c)+(-c) \implies a+[c+(-c)] = b + [c + (-c)] \implies a + 0 = b + 0 \implies a = b$.

(ii) $a \cdot 0 = 0, \quad \forall a$;
Proof. We use A3 and DL (definition of additive identity 0, and distributive law of mult. over add.) to obtain $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, which gives $0 + a \cdot 0 = a \cdot 0 + a \cdot 0$. From (i) above, we conclude $0 = a \cdot 0$ for all a.

(iii) $(-a)b = -ab, \quad \forall a, b$;
Proof. Since $a + (-a) = 0$ by definition of additive identity, we have $ab + (-a)b = [a + (-a)] \cdot b = 0 \cdot b = 0 = ab + (-(ab))$. Again from (i), we have $ab + (-a)b = ab + (-(ab)) \implies (-a)b = -(ab)$ as desired.

(iv) $(-a)(-b) = ab, \quad \forall a, b$;
Proof. Assume $a, b \in \mathbb{F}$. M2 gives commutativity of multiplication ($ab = ba \quad \forall a, b$). We have shown in (iii) that $(-a)b = -ab$, so consider:

$$\begin{aligned} (-a)(-b) &= -[a(-b)] \quad \text{(by (iii) above and brackets from M1 assoc. of mult.)} \\ &= -[(-b)(a)] \quad \text{(M2, commutativity of mult.)} \\ &= -[-(b)(a)] \quad \text{(applying (iii) again inside brackets)} \\ &= - - [ba] = ab \quad \text{(lemma and M2 commut. of mult.)} \end{aligned}$$

Lemma: For $x \in \mathbb{F}$, we have $-(-x) = x$.
Proof: Consider $x = x \implies (-x) + x = 0 \implies -(-x) = x$ because additive inverse of $-x$ is $-(-x)$ as seen by $(-x) - (-x) = 0$. If we know the additive inverse is unique, seeing this is even easier.

(v) $ac = bc, c \neq 0 \implies a = b$;
Proof. Assume $c \neq 0$ and $a, b, c \in \mathbb{F}$. By M4, we have some c^{-1} with $cc^{-1} = 1$ because $c \neq 0$. Then consider:

$$\begin{aligned} ac &= bc \quad \text{(given)} \\ (ac)c^{-1} &= (bc)c^{-1} \quad \text{(left-multiplication by } c^{-1}\text{)} \\ a[cc^{-1}] &= b[cc^{-1}] \quad \text{(assoc. of mult.)} \\ a[1] &= b[1] \quad \text{(M4, multiplicative inverse)} \\ a &= b \quad \text{(definition of 1, mult. identity)} \end{aligned}$$

(vi) $ab = 0 \implies [(a = 0) \cup (b = 0)], \quad \forall a, b, c \in \mathbb{R}$. *Proof.* If $ab = 0$ and $b \neq 0$, then $0 = b^{-1} \cdot 0 = 0 \cdot b^{-1} = (ab) \cdot b^{-1} = a(bb^{-1}) = a \cdot 1 = a$. We showed if $b \neq 0$, then $a = 0$. By symmetry, the other way holds.

□

Problem 3.4. Prove (v) and (vii) of Theorem 3.2 (textbook page 16).

Theorem 3.2, Ross p.16 (Additional Consequences of an Ordered Field) For all $a, b, c \in \mathbb{R}$,
(i) $a \leq b \implies -b \leq -a$;
(ii) $a \leq b, c \leq 0 \implies bc \leq ac$;
(iii) $0 \leq a, 0 \leq b \implies 0 \leq ab$;
(iv) $0 \leq a^2, \quad \forall a$;
(v) $0 < 1$;
(vi) $0 < a \implies 0 < a^{-1}$;
(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$;

Solution. (i) $a \leq b \implies -b \leq -a$;
Proof. Suppose $a \leq b$. By O4 applied to $c = (-a) + (-b)$, we have

$$a + [(-a) + (-b)] \leq b + [(-a) + (-b)].$$

By a cancellation law, it follows that $-b \leq -a$, as desired.

(ii) $a \leq b, c \leq 0 \implies bc \leq ac$;
Proof. If $a \leq b$ and $c \leq 0$, then $0 \leq -c$ by (i). Now **O5** gives us $a \leq b, 0 \leq c \implies ac \leq bc$, so we have

$$\begin{aligned} a \leq b, 0 \leq -c \implies a(-c) &\leq b(-c) \quad \text{(from \textbf{O5} with } -c) \\ -ac &\leq -bc \\ [ac + bc] - ac &\leq [ac + bc] - bc \quad \text{(or by (i) above)} \\ bc &\leq ac, \end{aligned}$$

as desired.

(iii) $0 \leq a, 0 \leq b \implies 0 \leq ab$;
Proof. From **O5.** we have $a \leq b, 0 \leq c \implies ac \leq bc$. If we set $a = 0$, we have our desired statement, just with different variables.

(iv) $0 \leq a^2, \quad \forall a$;
Proof. This is the trivial inequality. Consider 3 cases. If $a = 0$, then $0 \leq 0^2 = a^2$, as desired. If $a > 0$, then $a^2 > 0$ by (iii). If $a < 0$, then $0 < -a$ by adding $(-a)$ to both sides. Then we apply similar logic to $(-a)$:

$$\begin{aligned} a^2 &= a \cdot a \quad \text{(definition of iterated power)} \\ &= (-a)(-a) \quad \text{(Theorem 3.1, (iv))} \\ &\geq 0 \quad \text{(by (iii) above)} \end{aligned}$$

Hence we have shown for all cases $0 < a, 0 = a, 0 > a$, $a^2 \geq 0$.

(v) $0 < 1$;
Proof.

$$\begin{aligned} 1 \cdot 1 = 1^2 &\geq 0 \quad \text{(by (iv))} \\ 1 &\geq 0 \quad (1 \cdot 1 = 1 \text{ by definition of mult. identity}) \end{aligned}$$

It remains to show the strict inequality; that is, $0 < 1$. We already have $0 \leq 1$, so suppose for contradiction that $0 = 1$. Then take some $x \neq 0$, and consider:

$$\begin{aligned} 0 &= x + (-x) \quad \text{(add. inverse)} \\ &= 1 \cdot x + (-x) \quad \text{(def. of mult. identity 1)} \\ &= 0 \cdot x + (-x) \quad \text{(substitution } 1 = 0) \\ &= -x \quad \text{(definition of 0 as add. identity)} \\ 0 + x &= -x + x \quad \text{(add } x \text{ to both sides)} \\ x &= 0 \quad \text{(definition of add. inverse and add. identity)} \end{aligned}$$

However, we fixed some $x \neq 0$, and we reach a contradiction, so we cannot have $0 = 1$, which was to be shown.

(vi) $0 < a \implies 0 < a^{-1}$;
Proof. We are given $0 < a$, but suppose $0 \not< a^{-1}$. Then $a^{-1} \leq 0$, and $0 \leq -a^{-1}$. Then by (iii), we have

$$0 \leq a(-a^{-1}) = -1,$$

so $1 \leq 0$, a direct contradiction to (iv) which we have just proven, independent of this claim in (vi).

(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$;
Proof. The compound inequality $0 < a < b$ gives us $a > 0, b > 0, b > a$. From (vi), we have $b > 0 \implies b^{-1} > 0$ and $a > 0 \implies a^{-1} > 0$. Then consider:

$$\begin{aligned} a &< b \quad \text{(given)} \\ aa^{-1} &< ba^{-1} \quad \text{(right-multiplying by } a^{-1} > 0) \\ b^{-1} \cdot 1 &< b^{-1} \cdot ba^{-1} \quad \text{(left-multiplying by } b^{-1} > 0) \\ b^{-1} &< a^{-1} \quad (b^{-1}b = 1) \end{aligned}$$

So we have $0 < a^{-1}$, $0 < b^{-1}$, and $b^{-1} < a^{-1}$. This is equivalent to $0 < b^{-1} < a^{-1}$, which is our desired implication.

□

Problem 3.6. (a) Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

Solution. We apply the triangle inequality twice (as opposed to checking 8 cases).
We know $|a + b| \leq |a| + |b|$. Consider then that

$$\begin{aligned} |a + b + c| &\leq \big||a + b| + c\big| \quad \text{(from } (a + b) \leq |a + b|) \\ &\leq |a + b| + |c| \quad \text{(triangular ineq. on } (a + b), c) \\ &\leq |a| + |b| + |c| \quad \text{(triangular ineq. on } a, b) \end{aligned}$$

□

(b) Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

Solution. Consider the set $U \subset \mathbb{N}$ of natural numbers n that satisfy this generalized triangular inequality. Surely, $1 \in U$ from the trivial $|a_1| \leq |a_1|$. Also, $2 \in U$ as proven in the triangular inequality. Additionally, $3 \in U$ from above.
Assume $k \in U$ so that we have

$$|a_1 + a_2 + \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k|.$$

Then similar to as we have done in (a), consider:

$$\begin{aligned} |a_1 + a_2 + \cdots + a_k + a_{k+1}| &\leq \big||a_1 + a_2 + \cdots + a_k| + a_{k+1}\big| \quad \text{(because } (a_1 + a_2 + \cdots + a_k) \leq |a_1 + a_2 + \cdots + a_k|) \\ &\leq |a_1 + a_2 + \cdots + a_k| + |a_{k+1}| \quad \text{(triangular ineq. on the two terms)} \\ &\leq |a_1| + |a_2| + \cdots + |a_k| + |a_{k+1}| \quad \text{(triangular ineq. on } a_1 + a_2 + \cdots + a_k) \end{aligned}$$

We have shown $k \in U \implies (k + 1) \in U$, so by induction we have $U = \mathbb{N}$ and we are done. □

Problem 3.7. (a) Show $|b| < a$ if and only if $-a < b < a$.

Solution. ($\implies$) First we prove the forward direction. Suppose we have $|b| < a$. If $b = 0$, then $a > 0 = |0|$, and $-a < 0$. So if $b = 0$, we have $-a < b < a$, as required.

If $0 < b$, then $|b| = b$, and we have $0 < b = |b| < a$. Because $0 < a$, then $-a < 0 < b$. Hence we have $-a < b < a$, as required.

If $b < 0$, then $b < 0 < |b| < a$. Because $0 < a$, then $-a < 0$. Then we have $-a < 0 < b < a$, as required.

($\impliedby$) Now we prove the backwards direction. Suppose we have $-a < b < a$. If $b = 0$, then we have $-a < 0 < a$ and thus $0 = |bb| < a$, as desired.

If $b < 0$, then we have $-a < b < 0 < a$, and thus $|b| < |-a| = a$, as required.

If $b > 0$, then we have $-a < 0 < b < a$, and thus $|b| = b < a$, as required. $\square$

(b) Show $|a - b| < c$ if and only if $(b - c) < a < (b + c)$.

Solution. ($\implies$) If $|a - b| < c$, then from (a) we have $-c < (a - b) < c$. Then adding b across the inequalities, we get: $(b - c) < a < (b + c)$, precisely as required.

($\impliedby$) If $(b - c) < a < (b + c)$, then subtracting b across the inequalities, we get

$$-c < (a - b) < c.$$

If $(a - b) = 0$, then $|a - b| = 0 < c$, as desired. If $(a - b) > 0$, then $(a - b) = |a - b| < c$, as required. If $(a - b) < 0$, then $|a - b| = -(a - b)$ and $(a - b) > -c \implies -(a - b) = |a - b| < c$, so $|a - b| < c$, as required. $\square$

Problem 4.3, 4.4. For each set below, give its supremum if it has one. Otherwise, write "NO sup." And 4.4, do this with infima (plural of infimum).

(d) $S_d := \{e, \pi\}$
$\sup S_d = \pi, \inf S_d = e$

(e) $S_e := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
$\sup S_e = 1, \inf S_d = 0$

(i) $S_i := \cap_{n=1}^{\infty} \left[-\frac{1}{n},\ 1 + \frac{1}{n}\right] = [-1, 2] \cap \left[-\frac{1}{2}, 1 + \frac{1}{2}\right] \cap \left[-\frac{1}{3}, 1 + \frac{1}{3}\right] \cap \cdots$
$\sup S_i = 1, \inf S_i = 0$

(k) $S_k := \left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\} = \{0, 2.5, 3 - \frac{1}{3}, 4.25, 5 - \frac{1}{5}, 6 + \frac{1}{6}, 7 - \frac{1}{7}, \dots\}$
NO sup , $\inf S_k = 0$

(r) $S_r := \cap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = (0, 2) \cap \left(\frac{1}{2}, \frac{3}{2}\right) \cap \left(\frac{2}{3}, \frac{4}{3}\right) \cap \cdots$
Notice that $1 \in S_r$; $\sup S_r = 1 = \inf S_r$

(u) $S_u := \left\{x^2 : x \in \mathbb{R}\right\}$
NO sup, $\inf S_u = 0$

Problem 4.9. Complete the proof that $\inf S = -\sup(-S)$ in Corollary 4.5 by proving assertions (1) and (2). [Ross, p. 23]

Solution. For completeness (dad-tier pun intended), we include the completeness axiom, its corollary, and the entire proof of the corollary. Proofs of assertions (1) and (2) follow the proof given by Ross included here.

Completeness Axiom
Every *nonempty subset* $S \subset \mathbb{R}$ that is *bounded above* has a least upper bound, namely $\sup S$.

Corollary 4.5
Every *nonempty* subset $S \subset \mathbb{R}$ that is *bounded below* has a greatest lower bound, namely $\inf S$.

Proof. Let $-S$ be the set $\{-s : s \in S\}$; $-S$ consists of the negatives of the numbers in S. Recall, from the hypothesis of our corollary, that S is bounded below, so there is an $m \in \mathbb{R}$ with $m \leq s, \forall s \in S$. This implies $-m \geq -s, \forall s \in S$, so $-m \geq u, \forall u \in -S$. Thus $-S$ is bounded above by $-m$. The Completeness Axiom applies to $-S$, so $\sup(-S)$ exists. Intuitively, we prove $\inf S = -\sup(-S)$.

Let $s_0 := \sup(-S)$; we need to prove

$$-s_0 \leq s, \quad \forall s \in S, \tag{1}$$
$$(t \leq s, \quad \forall s \in S) \implies (t \leq -s_0). \tag{2}$$

The inequality (1) shows $-s_0$ is a lower bound for S, while (2) shows $-s_0$ is *the greatest* lower bound; that is, $-s_0 = \inf S$.

To prove (1), recall that $s_0 := \sup(-S)$ is the least upper bound of $-S$, so $\forall s \in S$, we have: $-s \leq \sup -S = s_0$. Then by adding $(s - s_0)$ to both sides of inequality $-s \leq s_0$, we get, for all $s \in S$:

$$-s + [s - s_0] \leq s_0 + [s - s_0] \implies -s_0 \leq s,$$

which gives (1) precisely.

To prove (2), similarly recall that s_0 is the least upper bound of $-S$. Then any upper bound of $-S$, say u with $-s \leq u, \forall s \in S$ cannot be any less than the least upper bound for $-S$. That is, $(-s \leq u, \forall s \in S) \implies (s_0 \leq u)$. Consider the substitution or definition $t := -u$, which is legal because we take u to be any upper bound of $-S$, so t is simply the negative of the value of u. Our implication reads:

$$\begin{aligned}(-s \leq u, \forall s \in S) &\implies (s_0 \leq u) \quad \text{(above)}\\ (-s \leq -t, \forall s \in S) &\implies (s_0 \leq -t) \quad \text{(substitution } t = -u)\\ (t \leq s, \forall s \in S) &\implies (t \leq -s_0) \quad \text{(consequences of field axioms)},\end{aligned}$$

which is (2) which was to be shown. □

Problem 4.12. Let $\mathbb{I}$ be the set of real numbers that are NOT rational; elements of $\mathbb{I}$ are called irrational numbers. Prove that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. (Hint: First show that $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset \mathbb{I}$.)

Thm. 4.7 Denseness of $\mathbb{Q}$: If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ with $a < r < b$.

Solution. From the Denseness of $\mathbb{Q}$ theorem, we already have the existence of some r with $a < r < b$. To show there is some $x \in \mathbb{I}$ with a similar property, we follow the provided hint and first show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset \mathbb{I}$. For contradiction, suppose there is some $\xi \in \{r + \sqrt{2}\}$ but $\xi \notin \mathbb{I}$. By our definition of $\mathbb{I}$, this ξ then must be rational. However, by $\xi \in \{r + \sqrt{2} : r \in \mathbb{Q}\}$, ξ must be the sum of some $r \in \mathbb{Q}$ and $\sqrt{2}$. Consider the case where $r = 0$. Then $\xi = \sqrt{2}$ is rational (as it is not in $\mathbb{I}$), which is a contradiction to a previously proven fact that $\sqrt{2}$ is rational. So we have $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset \mathbb{I}$ as suggested by the hint.

To show that there exists some $x \in (a, b)$ with $x \in \mathbb{I}$, consider for some $c, d \in \mathbb{R}$, that some r with $c < r < d$ is guaranteed by the Denseness of $\mathbb{Q}$. Adding $\sqrt{2}$ across the inequalities, we get: $c + \sqrt{2} < r + \sqrt{2} < d + \sqrt{2}$. Surely, $c + \sqrt{2}$ and $d + \sqrt{2}$ both live in $\mathbb{R}$, so we can define $a := c + \sqrt{2}$ and $b := d + \sqrt{2}$. And surely, $r + \sqrt{2} \in \{r + \sqrt{2} : r \in \mathbb{Q}\}$ by definition of the set. Then for some $\xi \in \{r + \sqrt{2} : r \in \mathbb{Q}\} \subset \mathbb{I}$, we have: $a < \xi < b$, as required. □

Problem 4.14. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) Prove $\sup(A + B) = \sup A + \sup B$. (Hint: To show $\sup A + \sup B \leq \sup(A + B)$, show that for each $b \in B, \sup(A+B)-b$ is an upper bound for A, hence $\sup A \leq \sup(A+B)-b$. Then show $\sup(A+B)-\sup A$ is an upper bound for B.)

(b) Prove $\inf(A + B) = \inf A + \inf B$.

Solution. (a) If we show $\sup A + \sup B \leq \sup(A + B) \leq \sup A + \sup B$, then we have equality $\sup(A + B) = \sup A + \sup B$.

[**Proving** $\sup A + \sup B \leq \sup(A + B)$]**:**

We claim that for some $b \in B$, $\sup(A + B) - b$ is an upper bound for A. For contradiction, suppose this is false. Then for each $b \in B$, there would exist some $a \in A$ with $a > \sup(A + B) - b$. Adding b to both sides of the inequality, this gives $a+b > \sup(A+B)$, which directly contradicts the definition of $\sup(A+B)$ as an upper bound of all such values of $(a + b)$. Thus we have that $\sup(A + B) - b$ is an upper bound for A, so by definition of $\sup A$, we have:

$$\begin{aligned} \sup A &\leq \sup(A + B) - b \quad \text{(as we just proved)} \\ \sup(A + B) - \sup B &\leq \sup(A + B) - b \quad (\sup B \in B, \text{ RHS always UB for } \sup A.) \\ \implies \sup(A) &\leq \sup(A + B) - \sup(B) \leq \sup(A + B) - b \quad (\text{so ineq holds for } b := \sup B) \end{aligned}$$

And adding $\sup B$ across the inequalities, we get:

$$\sup A + \sup B \leq \sup(A + B)$$

[**Proving** $\sup(A + B) \leq \sup A + \sup B$]**:**

Analogously, let us propose that $\sup A + \sup B - b$ is an upper bound for A. To see this, suppose not. Then

$$\exists_{b \in B}, \exists_{a \in A}[a > \sup A + \sup B - b] \implies [a + b > \sup A + \sup B].$$

However, we know by definition that $\forall a \in A, a \leq \sup A$ and $\forall b \in B, b \leq \sup B$. Adding these two inequalities give $a + b \leq \sup A + \sup B$, a direct contradiction to our finding that $a + b > \sup A + \sup B$. So our supposition cannot be correct, and we must have $\sup A + \sup B - b$ is an upper bound for A. And by definition of $\sup A$ as the least upper bound, for all $a \in A$ and $b \in B$, we have:

$$a \leq \sup A \leq \sup A + \sup B - b \implies a + b \leq \sup A + \sup B$$

And because this is true for all $a \in A, b \in B$, then we have

$$\sup(A + B) \leq \sup A + \sup B.$$

As mentioned above, because we have $\sup A+\sup B \leq \sup(A+B) \leq \sup A+\sup B$, we must have $\sup(A+B) = \sup A + \sup B$, which was to be shown.

□

Solution. (b) Now we prove $\inf(A+B) = \inf(A) + \inf(B)$. We similarly want to show $\inf(A+B) \leq \inf(A) + \inf(B) \leq \inf(A+B)$.

[**Proving** $\inf(A+B) \leq \inf(A) + \inf(B)$]

We claim that for each $b \in B, \inf(A+B) - b$ is a lower bound for A. For contradiction, suppose this is false. Then for some $b \in B$, we would have some $a \in A$ with $a < \inf(A+B) - b$, which by adding b across the inequality gives $a + b < \inf(A+B)$. Obviously this is not true (by definition that $\inf(A+B) \leq (a+b), \forall a \in A, b \in B$), so we have a contradiction. Thus we must have $\inf(A+B) - b$ as a lower bound for A.

Then we have, $\forall a \in A, b \in B$:

$$\inf(A+B) - b \leq \inf(A+B) - \inf(B) \leq \inf A \leq a$$

Adding $\inf(B)$ across the inequality $\inf(A+B) - \inf(B) \leq \inf(A)$, we get:

$$\inf(A+B) \leq \inf(A) + \inf(B).$$

[**Proving** $\inf(A) + \inf(B) \leq \inf(A+B)$]

Now we claim that for each $b \in B, \inf(A) + \inf(B) - b$ is a lower bound for A. For contradiction, suppose this is false. Then for some $b \in B$, we would have some $a \in A$ with $a < \inf(A) + \inf(B) - b$. Adding b across the inequality gives $a + b < \inf(A) + \inf(B)$, which again is a contradiction to the definition of infimum, because we can add the inequalities $\inf(A) \leq a$ and $\inf(B) \leq b$ to get $\inf(A) + \inf(B) \leq a + b$. So it must be so that $\inf(A) + \inf(B) - b$ is a lower bound for A, for all $b \in B$.

Thus we have, $\forall a \in A, b \in B$:

$$\inf(A) + \inf(B) - b \leq \inf(A) \leq a$$
$$\implies \inf(A) + \inf(B) \leq a + b$$

and because this is true for all a, b, it must hold for $(a+b) := \inf(A+B)$:

$$\inf(A) + \inf(B) \leq \inf(A+B).$$

We have shown $\inf(A) + \inf(B) \leq \inf(A+B) \leq \inf(A) + \inf(B)$, so we necessarily have $\inf(A) + \inf(B) = \inf(A+B)$, which was to be shown.

□

Problem 5.4. Show that $\inf S = -\sup(-S)$ in the case where S is not bounded below. [Note that you completed the proof of this statement when S is bounded below for problem 4.9 above, in proving the corollary of the Completeness Axiom, for a greatest lower bound.]

Solution. To show $\inf S = -\sup(-S)$, if we show $-\inf S = \sup(-S)$, we are done. Note that we avoid using the symbols $\pm\infty$ although the use of them would help illuminate the underlying concept. We are given that $S \subset \mathbb{R}$ is not bounded below. Then $\forall x \in \mathbb{R}$, there exists some $s \in S$ with $s < x$. We know this is equivalent to $-x < -s$ (or we can add $-s-x$ to both sides). Consider the set $-S$, with each element defined as the negative of each respective element in S. From our finding $-x < -s$, we have that for all numbers $-x \in \mathbb{R}$, there exists some element $-s \in -S$ which is greater than $-x$. Hence $-S$ is not bounded above, and $\sup(-S)$ is equivalent to $\inf(S)$, but in the positive direction as opposed to negative.

Due to the symmetry in our argument, we conclude that $-\inf S = \sup(-S)$. □

If we admit the use of $\pm\infty$, then we have:

1. (S is not bounded below) $\implies$ ($\inf S = -\infty \implies -[\inf(S)] = -(\infty) = \infty$), as permitted in lecture.
2. ($-S$ not bounded above) $\implies \sup(-S) = \infty$.

Hence $-\inf(S) = \infty = \sup(-S)$. Alternatively, we have $\inf(S) = -\infty$ and $\sup(-S) = \infty$. We have from lecture that $-(-\infty) = \infty$, and we are done.