Math 104, Summer 2019
Lecture 9, Monday 7/8/2019

CLASS ANNOUNCEMENTS: Today we'll be going over something very abstract, and it may not seem (immediately) applicable. Nothing today will be covered on Midterm 1.

Midterm covers lectures 1 7, in-class. Basically, no self-made cheat-sheets; the central theorems and results are required to be known by heart; so memorize those.

We shouldn't cite results non-assigned exercises in the book, but this won't be penalized (if those results help reduce a given problem).

As mentioned before, we'll have True-False sections questioning if a statement holds with slight modifications. We might have sections just asking to simply cite major results, like the definition of a Cauchy sequence or Convergence.

1 Topics/Goals:

- Metrics / Metric Spaces
- Completeness of $\mathbb{R}$

- Rudiments of Topology (in $\mathbb{R}$); generalization of "openness" and "closedness"
- Compactness (very fundamental to Topology)
- Heine-Borel Theorem (compact iff closed, bounded), a characterization of compactness in $\mathbb{R}^k$

For us, topological space will simply mean (in our case) metric space.

2 Metric Spaces

Definition: Metric Space -

A **metric space** is a set S together with a **metric** $d(\cdot,\cdot)$, a function defined for all pairs of elements $x, y \in S$. For our use, we want $d(x, y)$ to emulate the "distance" between x, y.

Axioms (Definitions) of a Metric Space:

D1. $d(x, x) = 0, \forall x \in S$ (distance from point to itself is 0, and nonzero distance implies distinct points)

D2. $d(x, y) = d(y, x), \forall x, y \in S$ (distance is symmetric in direction)

D3. $d(x, z) \le d(x, y) + d(y, z), \forall x, y, z \in S$ (triangle inequality)

Note that these three alone imply nonnegativity by setting $x = z$ and letting y be arbitrary.

Examples of metrics:

$$d(a,b) := |a-b|$$
$$d(\vec{x},\vec{y}) := \sqrt{(x_1-y_1)^2+\cdots+(x_k-y_k)^2}$$
$$d(\vec{x},\vec{y}) := |x_1-y_1|+|x_2-y_1|+\cdots+|x_k-y_k|, \qquad \text{taxicab metric}$$

Now we can generalize to an arbitrary metric space, by taking our definitions, and anywhere we have absolute values, we can just augment the statement into a metric space definition.
So, in order for a sequence to converge, we generalize past the $|s_n - s| < \epsilon$ definition to now say something about $d(s_n, s)$ and $d(s_n, s_m)$.

Definition: Convergence in a Metric Space -

A sequence (s_n) of points in a metric space S with a metric d **converges** to $s \in S$ if

$$d(s_n, s) \to 0 \qquad \text{in } \mathbb{R}$$

Remark: This isn't to say the distance is necessarily equal to 0; just that it tends to 0.

Definition: Cauchy Sequence -

We say (s_n) is **Cauchy** if :

$$\forall \epsilon > 0, \quad \exists_N \; [\; d(s_m, s_n) < \epsilon, \quad \forall m, n > N \;]$$

Definition: Completeness -

The (metric) space S is **complete** if **every** Cauchy sequence of points in S converges to some $s \in S$.

Remark: Thus we have a notion of completeness without ordering as for which we set axioms for $\mathbb{R}$.

So we may want to ask, does the Euclidean metric (as we normally conduct distance) give a complete space? We define

$$\mathbb{R}^k$$

to mean the Euclidean metric.

Lemma 2.1. A sequence $(\vec{x}^{(n)})$ converges in $\mathbb{R}^k$ if and only if each sequence $(x_j^{(n)})$ converges in $\mathbb{R}$ as $n \to \infty$.

That is, for each j, we have a sequence depending on n. Consider:

$$\begin{aligned}\vec{x}^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_k^{(1)})\\ \vec{x}^{(1)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_k^{(2)})\\ &\vdots\\ \vec{x}^{(1)} &= (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots, x_k^{(n)})\end{aligned}$$

So we construct:

$$\vec{x} = (x_1, x_2, \ldots, x_k).$$

We show:

$$\left| x_j^{(m)} - x_j^{(n)} \right| \leq d\left(\vec{x}^{(m)}, \vec{x}^{(n)}\right),$$

and also we know (as a generous bound):

$$d(\vec{x}^{(m)}, \vec{x}^{(n)}) \leq \sqrt{k} \max\left\{ \left| x_j^{(m)} - x_j^{(n)} \right| \right\}$$

Then as a theorem,

Theorem 2.2. Euclidean k-space $\mathbb{R}^k$ is **complete**.

To show this, we take an arbitrary Cauchy sequence and show it converges. By the above lemma, then each (x_j^n) is cauchy in $\mathbb{R}$ for each j. So $x_j^{(n)} \to x_j \in \mathbb{R}$ for some x_j as $n \to \infty$. And by the above lemma again, $\vec{x}^{(n)} \to \vec{x}$, where $\vec{x} = (x_1, \ldots, x_k)$.

3 Boundedness in $\mathbb{R}^k$

Definition: Bounded -

A set $S \subseteq \mathbb{R}^k$ is **bounded** if $\exists_M$ with

$$\max\{|x_j| : j = 1, \ldots, k\} \leq M, \qquad \forall \vec{x} \in S$$

Theorem 3.1. Bolzano-Weierstrass for $\mathbb{R}^k$:
Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

To see this, we start with sequence $(\vec{x}^{(n)})$, which is bounded. Then $(x_1^(n))$ is bounded as a sequence, so there exists some subsequence of $(\vec{x}^{(n)})$ such that $(s_1^{(n_k)})$ converges in $\mathbb{R}$. Then, we look at $(x_2^{n_k})$, and we take a convergent subsequence. We iterate through this a finite (k) number of times and construct our subsequence.
Essentially, we take our subsequences, drop terms until we get (or construct) something that converges.

4 Rudiments of Topology

Definition: Interior (or Neighborhood or Open Ball) -

Let $E \subseteq S$, where S is a metric space. We say $s_0 \in E$ is **interior** to E if $\exists_r \, [r > 0]$ such that:

$$(B_r(s_0) :=) \qquad \{s \in S : d(s, s_0) < r\} \leq E$$

Sometimes, there's no radius small enough that makes this work.
We define $E^o :=$ the set of interior points of E. And we say E is **open** if $E = E^o$ (i.e. every point of E is interior).

Definition 2.8 (Neighborhoods)**.** The *neighborhood* or *open ball* of a point x of radius $r > 0$ in a metric space is

$$N_r(x) := \{y \in X \mid d(x, y) < r\}$$

The *closed ball* of x and r is

$$\overline{N_r(x)} := \{y \in X \mid d(x, y) \leq r\}$$

It's often helpful to think of a metric space as a horribly twisted version of (a subset of) $\mathbf{R}^2$, so that it can be drawn on a piece of paper. Then, a ball $N_r(x)$ is literally a circle of radius r, centered on a point x.

13.8 Definition.
Let (S, d) be a metric space. A subset E of S is *closed* if its complement $S \setminus E$ is an open set. In other words, E is closed if $E = S \setminus U$ where U is an open set.

Because of (iii) in Discussion 13.7, the intersection of *any* collection of closed sets in closed [Exercise 13.5]. The *closure* E^- of a set E is the intersection of all closed sets containing E. The *boundary* of E is the set $E^- \setminus E^\circ$; points in this set are called *boundary points of E*.

To get a feel for these notions, we state some easy facts and leave the proofs as exercises.

13.9 Proposition.
Let E be a subset of a metric space (S, d).
- **(a)** *The set E is closed if and only if $E = E^-$.*
- **(b)** *The set E is closed if and only if it contains the limit of every convergent sequence of points in E.*
- **(c)** *An element is in E^- if and only if it is the limit of some sequence of points in E.*
- **(d)** *A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.*

Remark: The only sets in $\mathbb{R}$ that are both open and closed are: $\{\}$ and $\mathbb{R}$ itself.

Facts about Open Sets:

- For any (metric) space S, S is open in S, and $\{\}$ is open in S.
- Arbitrary (possibly infinite) unions of open sets are open
- **Finite** intersections of open sets are open

Considering the last point's restriction to finite intersections, note, as before we've found that

$$\cap_{n=1}^{\infty}\left(\frac{-1}{n}, 1+\frac{1}{n}\right) = [0,1].$$

Definition: Closed -

We say $E \subseteq S$ is **closed** if $S \backslash E$ is open. Additionally, we define:

$E^{-} :=$ the intersection of all closed subsets of S containing E ("closure of E").

So essentially, the closure of something like $[a, b)$ is effectively $[a, b]$. The closure of any set is unique.
Hence,

Definition: boundary -

The **boundary** of E is simply

$$E^{-} \backslash E^{o}.$$

Facts about Closed Sets:

- E is closed if and only if $E = E^{-}$.
- E is closed if and only if it contains the limit of every convergent sequence of points from E. (this is what we stated for the set of subsequential limits)
- A point s is in the bounary of E if and only if $s \in E$ and $s \in S \backslash E$.