Math 104, Summer 2019
Lecture 7, Wednesday 7/3/2019

CLASS ANNOUNCEMENTS: Tuesday in-class is a dedicated review session; bring questions! We'll probably have T/F questions, where correct answers benefit you, incorrect subtracts, and omit is neutral. Don't need to prove. Particular attention to 'is this statement still true when ...?' Heavy emphasis on generating counter-examples for these. The reason for this is that generating counter-examples is heavily intertwined with proving claims.

1 Review

Recall the following results from yesterday's lecture.

Theorem 1.1. (Ross 11.5); (Bolanzo-Weierstrass, for real sequences) Every bounded sequence has a convergent subsequence. Or in other words,

$$\text{bounded} \implies \text{exists convergent subsequence.}$$

Definition: Subsequential Limit -

A **subsequential limit** of a sequence (s_n) is any real number or $\pm\infty$ that is the limit of some subsequence of (s_n).

Theorem 1.2. (Ross 11.7) Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Theorem 1.3. (Ross 11.8) Let (s_n) be any sequence in $\mathbb{R}$, and let S denote its set of subsequential limits. We claim the following:

(i) S is nonempty
(ii) $\sup S = \lim s_n$, and $\inf S = \liminf s_n$.
(iii) $\exists \lim s_n \iff S = \{\lim s_n\}$, a singleton set.

2 Subsequential Limits

Theorem 2.1. (Ross 11.9) The set S of **subsequential limits** of a sequence (s_n) is "closed."
Or according to Ross, "Let S denote the set of subsequential limits of a sequence (s_n). Suppose (t_n) is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then $t \in S$."

Per our definition, we intersect S so that the terms themselves are not $\pm\infty$, although we allow the limits subsequential limits themselves to be $\pm\infty$.
The intuition behind this is that for:

$$t_1, t_2, t_3, \dots, \to t$$

we have some elements of subsequences:

$$\begin{aligned} &s_{n_1}^{(1)}, s_{n_2}^{(1)}, s_{n_3}^{(1)}, \dots, \to t_1 \\ &s_{n_1}^{(2)}, s_{n_2}^{(2)}, s_{n_3}^{(2)}, \dots, \to t_2 \\ &\vdots \end{aligned}$$

Do we have a subsequential limit $\to t$?

We can show $t \in S$ just be showing $(t-\epsilon, t+\epsilon)$ contains s_n for **infinitely many** n, for any $\epsilon > 0$.
Let $\epsilon > 0$. Since $t_n \to t$, we can find some **particular** n with

$$t_n \in (t-\epsilon, t+\epsilon).$$

THen choose δ so that

$$(t_n - \delta, t_n + \delta) \subseteq (t-\epsilon, t+\epsilon).$$

Consider $t_n \in S$, so it is the limit of (S_{n_k}), which is some subsequence of (s_n). Then we can find some N such that

$$\forall k > N, \qquad |s_{n_k} - t_n| < \delta.$$

Then all such $s_{n_k} \in (t-\epsilon, t+\epsilon)$. Intuitively, we force all of these to be within $\pm\delta$, tighter than within ϵ.
Or formally, from Ross:

Proof. Suppose t is finite. Consider the interval $(t-\epsilon, t+\epsilon)$. Then some t_n is in this interval. Let $\delta := \min\{t+\epsilon-t_n, t_n - t + \epsilon\}$, so that

$$(t_n - \delta, t_n + \delta) \subseteq (t-\epsilon, t+\epsilon).$$

Since t_n is a subsequential limit, the set $\{n \in \mathbb{N} : s_n \in (t_n - \delta, t_n + \delta)\}$ is infinite, so the set $\{n \in \mathbb{N} : s_n \in (t-\epsilon, t+\epsilon)\}$ is also infinite. Thus by Theorem 11.2(i), t itself is a subsequential limit of (s_n).
If $t = +\infty$, then clearly the sequence (s_n) is unbounded above, so a subsequence of (s_n) has limit $+\infty$ by Theorem 11.2(ii). Thus $+\infty$ is also in S. A symmetric argument applies if $t = -\infty$. □

3 lim sup, lim inf

We don't expect these to act like normal limits. We have the following by definition 10.6 and Thm 11.8:

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = \sup S \tag{1}$$
$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : N > N\} = \inf S. \tag{2}$$

Theorem 3.1. Let $s_n \to s > 0$, and (t_n) be any sequence. We claim

$$\limsup s_n t_n = s \cdot \limsup t_n$$

And conventionally, we say $s \cdot (\infty) = \infty, s \cdot (-\infty) = \infty$.

Proof. **Case 1:** Suppose $\limsup t_n = \beta \in \mathbb{R}$. We can take (t_{n_k}), a subsequence of (t_n) with $t_{n_k} \underset{k\to\infty}{\to} \beta$. Then we can find (s_{n_k}) such that it is a subsequence of (s_n) with $s_{n_k} \to s$. This shows that $s \cdot \beta$ is a subsequential limit of $s_n t_n$. That is,

$$s_{n_k} t_{n_k} \to s \cdot \beta.$$

Thus $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$. We conclude

$$\limsup s_n t_n \le s \cdot \beta.$$

Case 2: If $\limsup t_n = \infty$, then take (t_{n_k}) such that $t_{n_k} \to \infty$. Then $s_{n_k} \to s$. Hence $s_{n_k} t_{n_k} \to \infty$.
Case 3: If $\limsup t_n = -\infty$, then RHS $= -\infty$. So it's obvious that $\limsup s_n t_n \ge s \cdot \limsup t_n$.

Now to show:

$$\limsup s_n t_n \le s \cdot \limsup t_n$$

Assume WLOG that $s_n \neq 0$ for all n (by ignoring some finite number of terms at beginning of s_n). Then the sequence $\left(\frac{1}{s_n}\right)$ is defined, and moreover, by the limit theorems, we have

$$\lim \frac{1}{s_n} = \frac{1}{s}.$$

So we have

$$\limsup t_n = \limsup \left(\frac{1}{s_n}\right)(s_n t_n)$$

by the previous argument. t_n on the left is playing the role of $s_n t_n$, and $\left(\frac{1}{s_n}\right)$ is playing the role of s on the RHS of Case 1.
Precisely, we have:

$$\limsup t_n \ge \frac{1}{s} \limsup s_n t_n$$
$$s \cdot \limsup t_n \ge \limsup s_n t_n \quad \text{(we assumed } s > 0 \text{ at the start)}$$

□

Remark: It's important that we set in our assumption $s > 0$. If $s < 0$, we'd end up with some inf in our statement instead. If $s = 0$, then we would have another problem, because $0 \cdot (-\infty) = +\infty$ would not make sense. To see this, consider $s = 0$ with $s_n := \frac{-1}{n}, t_n := -n^2$. Then $s_n t_n = n$, and LHS $= \infty$ and RHS $= = 0 \cdot (-\infty)$.

3.1 What if $s = 0$ and (t_n) is bounded?

We're interested in what happens when we weaken or flex some of the hypotheses, as they should be minimal in theorems.

Theorem 3.2. (Ross 12.2) Let (s_n) be any sequence of **nonzero** real numbers. Then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Proof. The middle inequality follows easily. We prove the third inequality (and leave the first inequality to Exercise 12.11).
Let

$$\alpha := \limsup |s_n|^{1/n}, L := \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

That is, we will show $\alpha \le L$. To do this, we show $\alpha \le L_1,\ \forall L_1 > L$.
If $L = +\infty$, this is obvious, so we assume L is finite; that is, $L < +\infty$. Because

$$L = \limsup \left|\frac{s_{n+1}}{sn}\right| = \lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} < L_1,$$

we can find some term of the sequence (a positive integer N) with

$$\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \ge N\right\} < L_1$$

Hence

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1, \quad \forall n \ge N. \tag{3}$$

Then for $n > N$, we can write

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N|.$$

There are $n - (N+1) + 1 = n - N$ fractions, so applying (3), we have

$$|s_n| < L_1^{n-N}|s_N|, \quad \forall n > N.$$

Because L_1, N are fixed, $a := L_1^{-N}|s_N|$ is a positive constant and we may write:

$$|s_n| < {L_1}^n a, \quad \forall n > N \implies |s_n|^{1/n} < L_1 a^{1/n}, \quad \forall n > N.$$

Because $\lim_{n\to\infty} a^{1/n} = 1$ by Thm 9.7(d), we conclude that

$$\alpha = \limsup |s_n|^{1/n} \le L_1,$$

precisely as desired.

□

Corollary: If $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists, then

$$\lim |s_n|^{1/n} = L$$

exists.

Proof. If $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$, then all four values in the compound inequality from Ross 12.2 are equal to L. Hence $\lim |s_n|^{1/n} = L$, as desired. □

3.2 Example 2: How do s_n compare to $\limsup s_n$ and $\liminf s_n$?

(a) If $L := \limsup s_n \neq \infty$. then for every $\alpha > L$, the set $\{n : s_n > \alpha\}$ is **finite**. If $L \neq -\infty$, then for every $\beta < L$, the set $\{n : s_n > \beta\}$ is **infinite**. (Draw a picture on the real numberline to see this claim.)

Proof. If $L \neq \infty$ and $\alpha > L$, the set $\{n : s_n > \alpha\}$ is finite; otherwise $\sup\{s_n : n > N\} > \alpha$, for all N, and hence

$$L = \limsup s_n \geq \alpha > L,$$

a contradiction (see Def. 10.6). If $L \neq -\infty$ and $\beta < L$, the set $\{n : s_n > \beta\}$ is infinite; otherwise there exists a positive integer N_0 with $(n \geq N_0 \implies (s_n \leq \beta)$. Therefore, for all $N \geq N_0$,

$$\sup\{s_n : n > N\} \leq \beta$$

Hence

$$\limsup s_n \leq \beta < L,$$

a contradiction. The symmetric argument applies to (c). □

Remark: What happens when $\alpha = L$? Our argument for the above involves 'sliding' α left until it passes L.

(b) The set $\{n : s_n > \limsup s_n\}$ **can be** infinite. For example, $s_n = \frac{1}{n}$.
(c) If $L_0 = \liminf t_n \neq -\infty$, then the set $\{n : t_n < \beta_0\}$ is finite for $\beta_0 < L_0$. If $L_0 \neq \infty$, then the set $\{n : t_n < \alpha_0\}$ is infinite for $\alpha_0 > L_0$.
This follows from

$$L_0 = \liminf t_n = -\limsup(-t_n) = -L = -\limsup s_n,$$

where $s_n = -t_n$, and L is defined as in example (a) above. Then $(\beta_0 < L_0) \implies (-\beta_0 > -L_0 = L)$, so by (a) above,

$$\{n : t_n < \beta_0\} = \{n : -t_n > -\beta_0\} = \{n : s_n > -\beta_0\}$$

is finite. Similarly, $(\alpha_0 > L_0) \implies (-\alpha_0 < L)$, so

$$\{n : t_n < \alpha_0\} = \{n : -t_n > -\alpha_0\} = \{n : s_n > -\alpha_0\}$$

is **infinite**.
(d) If $\liminf s_n < \limsup s_n$, then the set

$$\{n : \liminf s_n \leq s_n \leq \limsup s_n\}$$

can be empty. For example, $s_n := (-1)^n(1 + \frac{1}{n})$.

Lecture ends here.

Food for thought: What does the set S of subsequent limits look like in the case (d) above, where there is **no** s_n within the interval $[\liminf s_n, \limsup s_n]$?