Math 104, Summer 2019
Lecture 6, Tuesday 7/2/2019

CLASS ANNOUNCEMENTS: Homework due Tuesday (not Monday), Exam Wednesday, extra OH on Tuesday.

1 Review: Cauchy Sequences

Definition: Cauchy Sequence (Ross p.62) -

A sequence (s_n) of real numbers is called a **Cauchy Sequence** if for each $\epsilon > 0$, there exists a number N with

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

Lemma 1.1. Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim s_n = s$. The idea is that because the terms s_n are close to s for large n, they also must be close to each other. That is,

$$|s_n - s_m| = |s_n - s + s - s_m| \le |s_n - s| + |s - s_m|.$$

To be precise, let $\epsilon > 0$. Then there exists N with $(n > N) \implies |s_n - s| < \frac{\epsilon}{2}$. Correspondingly, we may also write $(m > N) \implies |s_m - s| < \frac{\epsilon}{2}$. So we have

$$(m, n > N) \implies |s_n - s_m| \le |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence (s_n) is a Cauchy sequence. □

Lemma 1.2. Cauchy sequences are bounded.

Proof. Applying definition (10.8) of a Cauchy sequence, we have $N \in \mathbb{N}$ with

$$(m, n > N) \implies |s_n - s_m| < 1.$$

Because this is true for all n, m, it certainly holds for $n, N+1$ as follows: $|s_n - s_{N+1}| < 1$ for $n > N$. So $|s_n| < |s_{N+1}| + 1$, for $n > N$.
If $M = \max\{|s_{N+1}|+1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| \le M$, for all $n \in \mathbb{N}$. □

Remark: The following theorem on the next page is important, as it suffices to check if a sequence is Cauchy in order to verify that it converges. This does not involve the limit.

2 Proving Convergence $\iff$ Cauchy

Theorem 2.1. (Ross 10.11) A sequence is **convergent** if and only if it is **Cauchy**.

($\implies$) This was already proved in a previous lemma. That is, we verified that convergent sequences are Cauchy.
($\impliedby$) To check (ii), consider a Cauchy sequence (s_n) and note (s_n) is bounded by an above lemma. By theorem (10.7) (regarding $\liminf s_n = \lim s_n = \limsup s_n$), we need only show

$$\liminf s_n = \limsup s_n.$$

To see this, let $\epsilon > 0$. Because (s_n) is a Cauchy sequence, there exists N with

$$(m, n > N) \implies |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n : n > N\}$, so $v_N = \sup\{s_n : n > N\} \leq s_m + \epsilon$ for $m > N$. This shows $(v_N - \epsilon)$ is a lower bound for $\{s_m : m > N\}$, so

$$v_N - \epsilon \leq \inf\{s_m : m > N\} = u_N.$$

Thus we have:

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon.$$

Because this holds for all $\epsilon > 0$, we conclude that $\limsup s_n \leq \liminf s_n$. The opposite inequality always holds, as shown by $\liminf s_n = \limsup s_n$. We copy Ross 10.7 here for reference.

Theorem 2.2. (Ross 10.7)

(i) If $\lim s_n$ exists (even if $\pm\infty$), then

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n$, then s_n **converges** to their common value.

3 Subsequences; Subsequential Limits

Intuitively, a subsequence is basically just a selection of some [possibly all] of the original sequence, taken in the same order.

Definition: Subsequence -

Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A **subsequece** of this sequence is a sequence of the form $(t_k)_{k\in\mathbb{N}}$ where for each k, there is a positive integer n_k with

$$n_1 < n_2 < \cdots n_k < \cdots < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

Or in other words, to identify a subsequence, it suffices to give the original sequence and specify the indices that we want to select into our subsequence. In the above, n is the set of indices that are selected. We introduce the following function:

Definition: σ, Selection function -

We view the sequence $(s_n)_{n\in\mathbb{N}}$ as a function s with domain $\mathbb{N}$. For the subset $\{n_1, n_2, n_3, \dots\}$, there is a natural function σ given by

$$\sigma(k) := n_k, \qquad \text{for } k \in \mathbb{N}.$$

The function σ "selects" an infinite subset of $\mathbb{N}$, in order. The subsequence of s corresponding to σ is simply the composite function $t := s \circ \sigma$. That is,

$$t_k = t(k) = s \circ \sigma(k) = s[\sigma(k)] = s(n_k) = s_{n_k}, \qquad \text{for } k \in \mathbb{N}.$$

Thus a sequence t is a subsequence of a sequence s if and only if $t = s \circ \sigma$ for some increasing function σ mapping $\mathbb{N} \to \mathbb{N}$. We will usually suppress the notation σ and often suppress the notation t as well.

Example of subsequence: Say $a_n := \sin\left(\frac{n\pi}{3}\right)$. Listing out these terms, we have:

$$\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \frac{-\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \dots\right).$$

However, the subsequence of nonnegative terms is:

$$\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, 0, \dots\right).$$

We have $n_1 = 1; n_2 = 2; n_3 = 3; n_4 = 6; n_5 = 7; n_6 = 8, n_7 = 9, n_8 = 12, \dots$. We see that $n_k = 6k, s_{n_k} = 0$.

Theorem 3.1. (Ross 11.2) Let (s_n) be a sequence.

(i) If $t \in \mathbb{R}$, there is a subsequence converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ of indicies n is infinite, $\forall \epsilon > 0$.
(ii) If the sequence (s_n) is unbounded above, it has a subsequence with limit defined as $+\infty$.
(iii) Similarly, if (s_n) is unbounded below, it has a subsequence with limit $-\infty$.

Remark: In Ross we (can) suppose that any sequence has a monotonic subsequence. This is a consequence of the more general theorem Ross 11.4.

Proof. We can easily check the forward implications for (i), (ii), (iii). For example, if $\lim_k s_{n_k} = t$ and $\epsilon > 0$, then all but finitely many n_k's are in $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$. We focus on the other direction of implications.
(i) First suppose the set $\{n \in \mathbb{N} : s_n = t\}$ is infinite. Then there are subsequences (s_{n_k}) with $s_{n_k} = t$ for all k. Such subsequences of (s_n) are 'boring' monotonic sequences converging to t.

$$\vdots$$

□

Example : There exists a sequence (r_n) suhc that every rational number appears at least once.
By the denseness property of $\mathbb{Q}$, the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many rationals. So a is the limit of a subsequence of (r_n).

Theorem 3.2. (Ross 11.3) If the sequence (s_n) converges to s, then every subsequence converges to the same limit s.

Notice that the reverse direction is (quite) trivial. That is, if every subsequence of a sequence (s_n) converges to the same limit s, then the original sequence (s_n) converges to s.

Proof. Let (s_{n_k}) denote a subsequence of (s_n). Note that $n_k \geq k, \forall k$. This is easy to prove by induction; infact, $(n_1 \geq 1)$ and $(n_{k-1} \geq k-1)$ implies $n_k > n_{k-1} \geq k - 1$, and hence $n_k \geq k$.
Let $\epsilon > 0$. There exists N so that $n > N$ implies $|s_n - s| < \epsilon$. Now $k > N$ implies $n_k > N$, which implies $|s_{n_k}| < \epsilon$, so we write

$$\lim_{k\to\infty} s_{n_k} = s.$$

□

Theorem 3.3. (Ross 11.4) Every sequence (s_n) has a monotonic subsequence.

Proof. Let's say that the n-th term is **dominant** if it is greater than every term which follows it; that is,

$$(s_m < s_n), \qquad \forall (m > n).$$

Case 1: Suppose there are **infinitely many** dominant terms, and let (s_{n_k}) be any subsequence consisting solely of dominant terms. Then $s_{n_{k+1}} < s_{n_k}$ for all k by (1), so we conclude (s_{n_k}) is a decreasing sequence.
Case 2: Now suppose there are only finitely many dominant terms. Select n_1 so that s_{n_1} is beyond all the dominant terms of the sequence. Then given $(N \geq n_1)$,

$$\exists_{m>N} \left[s_m \geq s_N \right].$$

Applying this with $N := n_1$, we select $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Then s_{n_1} is not dominant. Suppose $n_1, n_2, \ldots, n_{k-1}$ have been selected so that

$$n_1 < n_2 < \cdots < n_{k-1},$$

and

$$s_{n_1} \leq s_{n_2} \leq \cdots \leq s_{n_{k-1}}.$$

Applying $[(N \geq n_1) \implies \exists_{m>N} \left[s_m \geq s_N \right]]$ with $N := n_{k-1}$, we select $n_k > n_{k-1}$ such that

$$s_{n_k} \geq s_{n_{k-1}}.$$

Then $n_1 < n_2 < \cdots < n_{k-1}$ and $s_{n_1} \leq s_{n_2} \leq \cdots \leq s_{n_{k-1}}$ hold with k in place of $k-1$.
We can formalize this procedure (algorithm) by induction. We conclude that we obtain an increasing subsequence (s_{n_k}). □

Remark: The following is incredibly important.

Theorem 3.4. (Ross 11.5); (Bolanzo-Weierstrass, for real sequences)
Every bounded sequence has a convergent subsequence. Or in other words,

$$\text{bounded} \implies \text{exists convergent subsequence.}$$

Proof. If (s_n) is a bounded sequence, it has a monotonic subsequence by Thm 11.4, which converges by Thm 10.2, and we are done. □

Definition: Subsequential Limit -

A **subsequential limit** of a sequence (s_n) is any real number or $\pm\infty$ that is a limit of the original sequence (s_n).

For example, for a convergent sequence $s_n \to s$, the set of subsequential limits is $\{s\}$, a singleton set, which is an if an only if with convergence to a limit.

3.1 Examples

$$\begin{aligned} s_n &:= (-1)^n n^2 \\ a_n &:= \sin\left(\frac{n\pi}{3}\right) \\ r_n &:= \text{ enum. of, or listing of every } \mathbb{Q} \\ b_n &:= n\left(1 + (-1)^n\right) \end{aligned}$$

The subsequential limits of the above are:

$$\begin{aligned} s_n &: \{\pm\infty\} \\ a_n &: \left\{0, \pm\frac{\sqrt{3}}{2}\right\} \\ r_n &: \mathbb{R} \cup \{\pm\infty\} \\ b_n &: \{0, \infty\} \end{aligned}$$

We hinted before that the limit inferior and limit superior are somehow related to subsequential limits. Before we make a claim and prove it, we first show that lim sup and lim inf actually are subsequential limits.

Theorem 3.5. (Ross 11.7) Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Proof. If (s_n) is not bounded above, then there exists a subsequence with limit $+\infty = \limsup s_n$. We will show there are infinitely many s_n within $(t - \epsilon, t + \epsilon)$, for any $\epsilon > 0$. The symmetric argument applies for $-\infty = \liminf s_n$.
Then $\exists_{N_0} [\sup\{s_n : n > N\} < t + \epsilon], \forall N > N_0$. For contradiction, suppose there is only a finite number of s_n in $(t-\epsilon, t+\epsilon)$, then we can find $N_1 > N_0$ so that $s_n \leq t - \epsilon$, for $n > N_1$. Then $t - \epsilon \geq v_{N_1} \geq \limsup s_n$, a contradiction. Thus our supposition must be false, and we must have an infinite number of s_n in $(t - \epsilon, t + \epsilon)$. □

(Recall that v_{N_1} is a decreasing sequence.) This leads us to a nice summarizing statement (or claim) as follows:

Theorem 3.6. (Ross 11.8) Let (s_n) be any sequence in $\mathbb{R}$, and let S denote its set of subsequential limits. We claim the following:

(i) S is nonempty
(ii) $\sup S = \lim s_n$, and $\inf S = \liminf s_n$.
(iii) $\exists \lim s_n \iff S = \{\lim s_n\}$, a singleton set.

Proof. (i) This is an immediate consequence of 11.7, where we have a monotonic subsequence (whose limit is $\limsup s_n$, and another with limit $\liminf s_n$).

(ii) Consider any limit t of a subsequence (S_{n_k}) of (s_n). Then by theorem 10.7, we have $t = \liminf_k (s_{n_k}) = \limsup_k (s_{n_k})$. Because $n_k \geq k, \forall k$, we have $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$, for each $N \in \mathbb{N}$.

Therefore,

$$\liminf_n s_n \leq \liminf_k s_{n_k} = t = \limsup_k s_{n_k} \leq \limsup_n s_n.$$

This inequality holds for all $t \in S$; therefore,

$$\liminf s_n \leq \inf S \leq \sup S \leq \sup s_n.$$

Therefore by Theorem 11.7, we have that $\liminf s_n$ and $\limsup s_n$ both belong to S.

(iii) This is simply a reformulation of Theorem 10.7, where $\exists \lim s_n \implies (\liminf s_n = \lim s_n = \limsup s_n)$, and that if $\liminf s_n = s = \limsup s_n$, then $s_n \to s$.

□

Remark: Hence if $\min S$ exists, it's $\inf S$. Analogously, if $\max S$ exists, it's $\sup S$.

Revisiting our examples,

$$\begin{aligned} s_n &:= (-1)^n n^2 \\ a_n &:= \sin\left(\frac{n\pi}{3}\right) \\ r_n &:= \text{ enum. of, or listing of every } \mathbb{Q} \\ b_n &:= n\left(1 + (-1)^n\right) \end{aligned}$$

The subsequential limits of the above are:

$$\begin{aligned} &s_n : \{\pm\infty\} : \liminf = -\infty; \limsup = +\infty \\ &a_n : \left\{0, \pm\frac{\sqrt{3}}{2}\right\} : \liminf = \frac{-\sqrt{3}}{2}; \limsup = \frac{\sqrt{3}}{2} \\ &r_n : \mathbb{R} \cup \{\pm\infty\} : \limsup = +\infty; \liminf = -\infty \\ &b_n : \{0, \infty\} : \limsup b_n = +\infty; \liminf b_n = 0 \end{aligned}$$