Math 104, Summer 2019
Lecture 5, Monday 7/1/2019

CLASS ANNOUNCEMENTS: Homework due tomorrow (Tuesday) at the beginning of class.

1 Review 2.9

We draw an example or visualization of what happens when we make the argument that a sequence converges. The idea is that if we track (and plot) the error: $|s_n - s|$ across n, then we see the error decays and approaches a small number (0). Moreover, we provide an upper bound (curve) above the $|s - s_n|$ plot for all n.

Definition 9.8: $+\infty$

For a sequence (s_n), we write $\lim s_n = \infty$ if for **any** $M > 0$, we can find N with $s_n > M$ for all integers $n > N$.

Definition 9.8: $-\infty$

For a sequence (s_n), we write $\lim s_n = -\infty$ if for **any** $M > 0$, we can find N with $s_n < M$ for all integers $n > N$.

Example 1:

$$\lim_{n\to\infty} \left(\sqrt{n} + 7\right) = \infty$$

Solution. Let $M > 0$. We want $\sqrt{n} + 7 > M \iff \sqrt{n} > M - 7 \iff n > \underbrace{(M-6)^2}_{=:N}$. □

Theorem 1.1. (Ross 9.9)
Let (s_n), (t_n) be sequences with $s_n \to \infty$, $t_n \to t > 0$. Then

$$s_n t_n \to \infty.$$

We can find N_1 with $t_n > \frac{t}{2}, \forall n > N_1$. For any M, take N_2 with $s_n > \frac{M}{t/2}, \forall n > N_2$. Then for $n > \max\{N_1, N_2\}$, we have $s_n t_n > \frac{M}{(t/2)}(t/2) = M$.

Theorem 1.2. (Ross 9.10) Let $s_n > 0, \forall n$.

$$\lim s_n = \infty \iff \lim \frac{1}{s_n} = 0$$

Proof. ($\implies$) Let $\epsilon > 0$. We want $\frac{1}{s_n} < \epsilon$, or equivalently, $s_n > \frac{1}{\epsilon}$. This can be achieved because $\lim s_n = \infty$.

($\impliedby$) Let $M > 0$. We want to show $s_n > M$. This is equivalent to $\frac{1}{s_n} < \frac{1}{M}$. This can be done because $\frac{1}{s_n} \to 0$. □

2 Monotonicity

Big Question: How can we tell that a sequence converges **without** referencing (or knowing) its limit?

How do we define e (or π)?
Perhaps commonly, $e := \lim \left(1 + \frac{1}{n}\right)^n$. Or perhaps $e := 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$.
There should be some other way we can define convergence rather than pointing out its limit. We do this by showing it is a Cauchy sequence. But before we get to this...

Definition: increasing sequence -

A sequence s_n is **increasing** (or non-decreasing) if $s_1 \leq s_2 \leq \cdots$, or equivalently,

$$\forall n, \qquad s_{n+1} \geq s_n.$$

Definition: decreasing sequence -

A sequence s_n is **decreasing** (or non-increasing) if $s_1 \geq s_2 \geq \cdots$, or equivalently,

$$\forall n, \qquad s_{n+1} \leq s_n.$$

Definition: monotonic -

We say (s_n) is **monotonic** if it is either increasing (nondecreasing) or decreasing (nonincreasing).

Examples of Monotonicity:

$$a_n = 1 - \frac{1}{n}$$
$$b_n = n^3$$
$$c_n = \left(1 + \frac{1}{n}\right)^n$$
$$d_n = \frac{1}{n^2}$$

Theorem 2.1. Bounded monotonic sequences converge.

Proof. WLOG take s_n to be monotonically increasing (our argument is symmetric for the monotonically decreasing case). Take $s := \sup\{s_n\}$. Let $\epsilon > 0$. Consider $s - \epsilon$ is not an upper bound (because otherwise s would not be least upper bound). Thus this implies there exists N with $s_N > s - \epsilon$, where $s_n > s - \epsilon, \quad \forall n > N$ because $s_n \geq S_N$. Then $(s - s_n) = |s - s_n| < \epsilon$. □

2.1 Examples of Applications of Ross 10.2

We write

$$K.d_1d_2d_3\ldots := K + \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_n}{10^n} + \cdots,$$

where $d_n \in \{0,1,2,3,\ldots,9\}$.
We can show, for instance,

$$\frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^n} < 1.$$

To see this, we can show that the partial sums

$$s_n := K + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}$$

is bounded above by $K+1$ (by definition of decimal representations).
We can also show

$$0.\overline{9} = 1,$$

so decimal expansions are NOT unique. For example, we can show that the Dedekind cuts $0.\overline{9}^* = 1^*$.

Remark: We can use (10.2), Bounded monotonic sequences converge) to analyze the recursively defined sequence

$$s_1 := 5; s_{n+1} := \frac{s_n^2+5}{2s_n}$$

Solution. Clearly $s_n > 0$. We want to show $\frac{s^2+5}{2s_n} = s_{n+1} \leq s_n$, which illustrates this sequence is monotonically incraeasing.
We can not only show that 0 is a lower bound, but we can also show all terms of the sequence are greater than or equal to something like $\sqrt{5}$ (see Ross). □

Theorem 2.2. (Ross 10.4) Unbounded increasing sequences diverge to $+\infty$; analogously, unbounded decreasing sequences diverge to $-\infty$.

Proof. For a sketch of the proof, if (s_n) is unbounded and increasing, it is bounded below by s_1, so it must be unbounded above. Then from the assumption of unbounded above, then $\forall M$, we can find N with $s_N > M$. Then $s_n \geq S_N > M$, for all $n > N$. □

3 Cauchy Sequences

The goal of a Cauchy Sequence is to recognize a converging sequence without using its limit. But before we get there...
For an **arbitrary bounded** sequence (s_n), limiting behavior depends only on sets of form

$$\{s_n | n > N\}.$$

We could ask: What is the supremum (or inf) of the set of all values? That is,

$$\sup\{s_n \text{ over all } n\}, \inf\{s_n \text{ over all } n\}$$

Definition: Limit superior, Limit inferior -

(For a bounded sequence s_n,)

$$\limsup s_n := \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$
$$\liminf s_n := \lim_{N\to\infty} \inf\{s_n \mid n > N\}$$

Also, as in Ross, we can define:

$$u_N := \inf\{s_n : n > N\}; \qquad v_N := \sup\{s_n : n > N\}$$
$$\liminf s_n := \lim_{N\to\infty} u_N; \qquad \limsup s_n := \lim_{N\to\infty} v_N$$

Notice that in our definition, any v_i describes the supremum of all v_n with $n > i$ (to the right of i).

Remark: If (s_n) is not bounded above, we say

$$\sup\{s_n : n > N\} := +\infty, \forall N;$$

$$\limsup s_n := +\infty.$$

Correspondingly, if (s_n) is not bounded below, we say

$$\inf\{s_n : n > N\} := -\infty, \forall N;$$

$$\liminf s_n := -\infty.$$

(In-class exercise:) Describe v_N for the following sequences:

$$s_n := (-1)^{n-1)}\left(1 - \frac{1}{n}\right)$$
$$s'_n := (-1)^{n-1}\left(1 + \frac{1}{n}\right)$$

Solution. My immediate approach would be to split n into odd and even cases. For n odd, we have:

$$s_n = \left(1 - \frac{1}{n}\right)$$
$$s'_n = \left(1 + \frac{1}{n}\right)$$

And when n is even, we have:

$$s_n = -\left(1 - \frac{1}{n}\right)$$
$$s'_n = -\left(1 + \frac{1}{n}\right)$$

□

Remark: We use the following theorem (Ross 10.7) to show that Cauchy $\implies$ Convergent in Lecture 6.

Theorem 3.1. (Ross 10.7)

(i) If $\lim s_n$ exists (even if $\pm\infty$), then

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n$, then s_n **converges** to their common value.

Proof. Claim (i): Suppose $\lim s_n = \infty$. Then for all $n > N$, $s_n > M$. Hence we have $u_m > M$ for $m > N$, where u_m is the infimum for the tail of the sequence. So $u_m \to \infty$, because we can do this for any arbitrary M (so we say $\liminf s_n = \infty$).
Suppose $s_n \to s \neq \pm\infty$. Then this gives us $|s_n - s| < \epsilon \implies -\epsilon < (s_n - s) < \epsilon$, which gives $s - \epsilon < s_n < s + \epsilon$, for all $n > N$.
Then $|u_m - s| < \epsilon$ and $|v_m - s| < \epsilon$ for ALL $m > N$, hence we have:

$$u_m \to s, \qquad v_m \to s$$

□

Proof. Claim (ii): Note that $u_m \leq s_n \leq v_n$, for all n. Note this does NOT follow directly from the squeeze lemma, as the squeeze lemma assumes that the sequence converges within the hypothesis.
If $\liminf s_n = \limsup s_n = +\infty$, it is easy to show $\lim s_n = +\infty$. Likewise, if $\liminf s_n = \limsup s_n = -\infty$, we also can easily show $\lim s_n = -\infty$.
Suppose that $\liminf s_n = \lim sups_n = s$, where $s \in \mathbb{R}$. We want to prove $\lim s_n = s$.

Let $\epsilon > 0$. Since $s = \lim v_N$, there exists some $N_0 \in \mathbb{Z}^+$ with

$$|s - \sup\{s_n : n > N_0\}| < \epsilon.$$

Thus $\sup\{s_n : n > N_0\} < s + \epsilon$, so

$$s_n < s + \epsilon, \forall n > N_0.$$

Similarly, there exists N_1 with

$$|s - \inf\{s_n : n > N_1\}| < \epsilon,$$

so $\inf\{s_n : n > N_1\} > s - \epsilon$, hence

$$s_n > s - \epsilon, \forall n > N_1.$$

From these two results, we conclude

$$s - \epsilon < s_n < s + \epsilon \implies |s_n - s| < \epsilon, \forall n > \max\{N_0, N_1\}.$$

This proves $\lim s_n = s$, as required. □

Definition: Cauchy sequence -

A sequence (s_n) is **Cauchy** if for any $\epsilon > 0$, we can find N with $|s_n - s_m| < \epsilon$, for all $n, m > N$. In other words, any two terms of the sequence is within this distance of each other.

A flawed version of this might be trying to define this as $|s_{n+1} - s_n| < \epsilon$, but this may still diverge.

Lecture ends here.

Remark: We will prove tomorrow that a sequence is convergent if and only if it is Cauchy.

Definition: Cauchy Sequence (Ross p.62) -

A sequence (s_n) of real numbers is called a **Cauchy Sequence** if for each $\epsilon > 0$, there exists a number N with

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

Lemma 3.2. Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim s_n = s$. The idea is that because the terms s_n are close to s for large n, they also must be close to each other. That is,

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|.$$

To be precise, let $\epsilon > 0$. Then there exists N with $(n > N) \implies |s_n - s| < \frac{\epsilon}{2}$. Correspondingly, we may also write$(m > N) \implies |s_m - s| < \frac{\epsilon}{2}$. So we have

$$(m, n > N) \implies |s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence (s_n) is a Cauchy sequence. □

Lemma 3.3. Cauchy sequences are bounded.

Proof. Applying definition (10.8) of a Cauchy sequence, we have $N \in \mathbb{N}$ with

$$(m, n > N) \implies |s_n - s_m| < 1.$$

In particular, $|s_n - s_{N+1}| < 1$ for $n > N$. So $|s_n| < |s_{N+1}| + 1$, for $n > N$. If $M = \max\{|s_{N+1}|+1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| \leq M$, for all $n \in \mathbb{N}$. □

Remark: The following theorem is important, as it suffices to check if a sequence is Cauchy in order to verify that it converges. This does not involve the limit.

Theorem 3.4. (Ross 10.11) A sequence is **convergent** if and only if it is **Cauchy**.

($\implies$) This was already proved in a previous lemma. That is, we verified that convergent sequences are Cauchy.
($\impliedby$) To check (ii), consider a Cauchy sequence (s_n) and note (s_n) is bounded by an above lemma. By theorem (10.7) (regarding $\liminf s_n = \lim s_n = \limsup s_n$), we need only show

$$\liminf s_n = \limsup s_n.$$

To see this, let $\epsilon > 0$. Because (s_n) is a Cauchy sequence, there exists N with

$$(m, n > N) \implies |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n : n > N\}$, so $v_N = \sup\{s_n : n > N\} \leq s_m + \epsilon$ for $m > N$. This shows $(v_N - \epsilon)$ is a lower bound for $\{s_m : m > N\}$, so

$$v_N - \epsilon \leq \inf\{s_m : m > N\} = u_N.$$

Thus we have:

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon.$$

Because this holds for all $\epsilon > 0$, we conclude that $\limsup s_n \leq \liminf s_n$. The opposite inequality always holds, as shown by $\liminf s_n = \limsup s_n$.