Math 104, Summer 2019
Lecture 4, Thursday 6/27/2019

CLASS ANNOUNCEMENTS: Homework 2 is due next **at the beginning of class** on Tuesday (as opposed to Monday). We start off by looking at proof-writing techniques via the textbook examples. Look for similarities in the structure of our formal proofs, and pay special attention to tricks we use to achieve these steps.

1 Textbook Examples of Limit Proofs

Example 1: Prove $\lim \frac{1}{n^2} = 0$. (went over this example in class)

Scratch work:
Our task is to consider an arbitrary $\epsilon > 0$ and show there exists a number N (which will depend on ϵ) such that $n > N \implies \left|\frac{1}{n^2} - 0\right| < \epsilon$. So we expect our formal proof to begin with "Let $\epsilon > 0$" and to end with something like "Hence $n > N \implies \left|\frac{1}{n^2} - 0\right| < \epsilon$". The content of the proof should specify an N and then verify N has the desired property for the arbitrary ϵ (namely the implication).
Similar to how we work with trigonomic identities, we initially reverse-engineer backwards from the conclusion, but in our "formal proof" we need to be sure that our steps are reversible. For this example, we want $\left|\frac{1}{n^2} - 0\right| < \epsilon$ and we want to know how big n must be. So operating on this inequality, we can effectively "solve" for n. After manipulations, we see that we want $n > \frac{1}{\sqrt{\epsilon}}$.
If our steps are reversible, then we have that

$$(n > \frac{1}{\sqrt{\epsilon}}) \implies \left|\frac{1}{n^2} - 0\right| < \epsilon.$$

So we assign $N := \frac{1}{\sqrt{\epsilon}}$.

Formal Proof. Let $\epsilon > 0$, $N := \frac{1}{\sqrt{\epsilon}}$. Then $(n > N) \implies (n > \frac{1}{\sqrt{\epsilon}})$, which implies $n^2 > \frac{1}{\epsilon}$, and hence $\epsilon > \frac{1}{n^2}$. We have shown $(n > N) \implies (\left|\frac{1}{n^2} - 0\right| < \epsilon)$, so $\lim \frac{1}{n^2} = 0$ and we are done. □

Example 2: Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.
Discussion: Notice that by context, it is clear that we want the limit as $n \to \infty$. We want

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon \equiv \left|\frac{19}{7(7n-4)}\right| < \epsilon.$$

Because $(7n-4) > 0$ for $n > N$, we can drop the absolute value and proceed:

$$\frac{19}{7\epsilon} < 7n - 4 \equiv n > \frac{19}{49\epsilon} + \frac{4}{7}$$

Our steps are reversible, so we assign $N := \frac{19}{49\epsilon} + \frac{4}{7}$. Incidentally, we could have chosen N to be any number larger than this.

This last sentence is important!
We only need to show that there is convergence at some point, which means we just need to show that past some finite N, $n > N$ gives convergence. Because of this, we can use ridiculous bounds and assumptions like as Example 3.

Formal Proof:
Let $\epsilon > 0$ and let $N := \frac{19}{49\epsilon} + \frac{4}{7}$. Then $(n > N) \implies \left(n > \frac{19}{49\epsilon} + \frac{4}{7}\right)$, and equivalently, this last inequality gives $\frac{19}{7(7n-4)} < \epsilon$.
Thus $\left|\frac{3n+1}{7n-4}\right| < \epsilon$, proving that $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

NOTE: The 'scratch work' is omitted in some of the following examples. They're included in Ross p.41 if you want to check your work, but try to guess how the resulting proofs make sense.

These 'formal proofs' will often look like they're pulled out of a bunny's behind that was pulled out of a hat, but these proofs show how *minimal* and *concise* a formal proof can be, where you hide all your scratch work and just let the reader or grader work out the steps.

Example 3: Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$.

Proof. Let $\epsilon > 0$, $N := \max\left\{2, \sqrt{\frac{52}{\epsilon}}\right\}$. Then $(n > N) \implies (n > \sqrt{\frac{54}{\epsilon}})$, which equivalently gives $\frac{27n}{n^3/2} < \epsilon$. Since $n > 2$, we have $\frac{n^3}{2} \leq n^3 - 6$ and also $27n \geq 3n + 24$. Thus $n > N$ implies:

$$\frac{3n+24}{n^3-6} \leq \frac{27n}{n^3/2} = \frac{54}{n^2} < \epsilon,$$

so equivalently this gives:

$$\left|\frac{4n^3+3n}{n^3-6} - 4\right| < \epsilon,$$

as required to show $\lim \frac{4n^3+3n}{n^3-6} = 4$. □

Remark: We went over this example in class, using the trick

$$n^3 - 6 > n^3/2, \text{ for } n > 2.$$

Then to catch the $n > 2$ assertion, we insert $N := \max\{2, \text{something}\}$. Max gave us really important advice that if we have a **finite** amount of conditions (i.e. $>$) to satisfy, we can simply assert something like the above.

It's also important to note that we could've 'invented' any trick different from the above, as long as we preserve the 'dominant' characteristic, which in the case was the exponent 3 in $n^3 - 6$ and $n^3/2$.

This can get difficult and a bit tricky, whereas with the Limit Theorems covered in the next section of this document (and Ross 1.9) , we could just write:

$$\lim\left[\frac{4n^3+3n}{n^3-6}\right] = \lim\left[\frac{4+\frac{3}{n^2}}{1-\frac{6}{n^3}}\right] = \frac{\lim 4 + 3\cdot\lim\left(\frac{1}{n^2}\right)}{\lim 1 - 6\cdot\lim\left(\frac{1}{n^3}\right)} = 4$$

But for now, we don't have that (and we want to understand formal proofs using definitions underlying limits and convergence).

Example 4: Show that the sequence $a_n = (-1)^n$ does not converge.

Formal Proof: Assume for contradiction that $\lim(-1)^n = a$ for some $a \in \mathbb{R}$. Defining $\epsilon := 1$ in the definition of limit, we see there exists N with

$$(n > N) \implies (|(-1)^n - a| < 1).$$

By considering both an even n and odd n greater than N, we have:

$$|1-a| < 1 \quad \text{and} \quad |-1-a| < 1.$$

Then by the Triangle Inequality, we have:

$$2 = |1-(-1)| = |1-a+a-(-1)| \leq |1-a| + |a-(-1)| < 1+1$$

But surely our finding $2 < 2$ is false, so our assumption that $\lim(-1)^n = a$ must be wrong, and hence our sequence $(-1)^n$ does not converge (and does not have a limit). $\square$

Example 5: Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note that $s \geq 0$ (exercise 8.9). Prove $\lim \sqrt{s_n} = \sqrt{s}$.

Scratch work: Just as we have earlier, we let $\epsilon > 0$ and show there exists some N with $n > N \implies |\sqrt{s_n} - \sqrt{s}| < \epsilon$.
But this time we cannot expect to obtain N explicitly in terms of ϵ because of the general (non-specific) nature of the problem. However, we can hope to show such N exists. The trick here is to violate our training in algebra and "irrationalize the denominator" (remember this 'trick'; it seems useful):

$$\sqrt{s_n} - \sqrt{s} = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}.$$

Since $s_n \to s$, we will be able to make the numerator small (for large n). Unfortunately for us, if $s = 0$, the denominator will also be small. So we split into two cases:
If $s > 0$ ($s \neq 0$), the denominator is bounded below by $\sqrt{s}$ and our trick will work:

$$|\sqrt{s_n} - \sqrt{s}| \leq \frac{|s_n - s|}{\sqrt{s}},$$

so we select N so that $|s_n - s| < \sqrt{s}\epsilon$ for $n > N$. Note that N exists, since we can apply the definiton of limit to $\sqrt{s}\epsilon$ just as well as to ϵ.
If $s = 0$, it can be shown directly that $\lim s_n = 0$ implies $\lim \sqrt{s_n} = 0$; our trick of "irrationalizing the denominator" is not necessary for this case.

Formal Proof: Case I: $s > 0$: Let $\epsilon > 0$. Since $\lim s_n = s$, there exists N with ($n > N \implies |s_n - s| < \sqrt{s}\epsilon$). Consider then that $n > N$ implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon.$$

Case II: $s = 0$. The proof to this is left as an exercise (Exercise 8.3). We solve this in class. With $s = 0$ in this case, we then have $s_n \to 0$. That is, we want to prove that if $s_n \geq 0$ and $s_n \to 0$, then $\sqrt{s_n} \to 0$.
Let $\epsilon > 0$. Since $\lim s_n = 0$, then there exists N with

$$(n > N \implies |s_n - 0| = |s_n| < \epsilon^2)$$

□

Example 6: Let (s_n) be a convergent sequence of real numbers with $s_n \neq 0$, for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove that

$$\inf\{|s_n| : n \in \mathbb{N}\} > 0.$$

Scratch work: The idea is that "most" of the terms s_n are close to s and hence are not close to 0. More explicitly, most terms s_n are within $\frac{1}{2}|s|$ of s, and thus most s_n satisfy $|s_n| \geq \frac{1}{2}|s|$. This seems clear from a picture argument (figure 8.1, Ross p. 55); however, a formal proof as below will use the triangle inequality.

Formal Proof: Let $\epsilon = \frac{1}{2}|s| > 0$. Since $\lim s_n = s$, there exists $N \in \mathbb{N}$ with ($n > N \implies |s_n - s| < \frac{|s|}{2}$). We claim

$$n > N \implies |s_n| \geq \frac{|s|}{2}. \tag{1}$$

To see this, suppose otherwise, and the triangle inequality would give

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|,$$

which is obviously false.
Let us define

$$m := \left\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\right\},$$

and we clearly have $m > 0$ and $|s_n| \geq m, \forall n \in \mathbb{N}$, in view of our claim (1). Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$, as desired. □

Remark: This concludes the set of examples provided in Ross. Formal proofs are expected for this sort of problems.

2 Limit Theorems for Sequences

First we prove convergent sequences are bounded. We define that a sequence (s_n) of real numbers is said to be *bounded* if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set; that is, if there exists a constant M with $|s_n| \leq M, \forall n$.

Theorem 2.1. Convergent sequences are bounded.

Proof. Let (s_n) be a convergent sequence, and let $s := \lim s_n$. Applying the definition of convergence (7.1) with $\epsilon = 1$, we get $N \in \mathbb{N}$ with

$$n > N \implies |s_n - s| < 1.$$

From the triangle inequality, we see $n > N \implies |s_n| < |s| + 1$. We define

$$M := \max\{|s| + 1, |s_1|, |s_2|, \ldots, |s_N|\}.$$

Then we have $|s_n| \leq M, \forall n \in \mathbb{N}$, hence (s_n) is a bounded sequence as desired. □

Theorem 2.2. If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks. Or equivalently,

$$\lim(ks_n) = k \cdot \lim s_n.$$

Proof. If $k = 0$, our result is immediately true. Then suppose $k \neq 0$. Let $\epsilon > 0$ and note that we need to show $|ks_n - ks| < \epsilon$ for large n. Since $\lim s_n = s$, then there exists N with

$$(n > N \implies |s_n - s| < \frac{\epsilon}{|k|}).$$

Thus we have

$$(n > N \implies |ks_n - ks| < \epsilon).$$

□

Theorem 2.3. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$. In other words,

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Proof. Let $\epsilon > 0$; we need to show $|s_n + t_n - (s + t)| < \epsilon$, for large n. Notice $|s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t|$ by the triangle inequality. Since $\lim s_n = s$, there exists N_1 with $n > N_1 \implies |s_n - s| < \frac{\epsilon}{2}$. Similarly, there exists N_2 with $n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}$.
Let $N := \max\{N_1, N_2\}$. Then clearly, adding these inequalities and adjoining with the triangle inequality, we see :

$$n > N \implies |s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

□

Theorem 2.4. If (s_n) converges to s, and (t_n) converges to t, then (s_nt_n) converges to st. In other words,

$$\lim(s_nt_n) = (\lim s_n)(\lim t_n).$$

The trick here is within the inequality:

$$\begin{aligned} |s_nt_n - st| &= |s_nt_n - s_nt + s_nt - st| \\ &\leq |s_nt_n - s_nt| + |s_nt - st| \\ &= |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \end{aligned}$$

For large n, $|t_n - t|$ and $|s_n - s|$ are small, and t is constant. Theorem 3.1 (Ross 9.1) shows $|s_n|$ is bounded, as it converges, so we will be able to show that $|s_nt_n - s_nt|$ is small.

Proof. Let $\epsilon > 0$. We have shown before that a convergent sequence is bounded. Thus there is a constant $M > 0$ with $|s_n| \leq M, \forall n$. Because $\lim t_n = t$, we have some N_1 with

$$n > N_1 \implies |t_n - t| < \frac{\epsilon}{2M}.$$

Additionally, because $\lim s_n = s$, we have some N_2 with

$$n > N_2 \implies |s_n - s| < \frac{\epsilon}{2(|t|+1)}.$$

Note that we used $\frac{\epsilon}{2(|t|+1)}$ instead of $\frac{\epsilon}{2|t|}$ because t could take on 0.
Consider that if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$\begin{aligned} |s_nt_n - st| &\leq |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ &\leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Proof. Here is the proof presented in Lecture. The statement is: If $s_n \to s, t_n \to t$, then $s_nt_n \to st$. (Suppose $s, t > 0$).
Choose $\epsilon_1 < \min\left\{\frac{epsilon}{3t}, \sqrt{\frac{\epsilon}{3}}\right\}$ and $\epsilon_2 < \min\left\{\frac{\epsilon}{3s}, \sqrt{\frac{\epsilon}{3}}\right\}$.
Then take N_1 with $|s_n - s| < \epsilon_1$ and N_2 with $|t_n - t| < \epsilon_2$. Take $N > \max\{N_1, N_2\}$, and then $\forall n, |s_nt_n - st| < \epsilon_1 t + \epsilon_2 s + \epsilon_1 \cdot \epsilon_2 < \epsilon$ due to the above choices. □

Theorem 2.5. Reciprocals: If (s_n) converges to s, if $s_n \neq 0, \forall n$, and if $s \neq 0$, then $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

Scratch work: Consider the equality:

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n \cdot s}\right|.$$

For large n, the numerator is small (given s_n converges to s). The only possible difficulty would be if the denominator were also small for large n.

We solved this difficulty in Example 6 in the Textbook Examples section above, where

$$m := \inf\{|s_n| : n \in \mathbb{N}\} > 0.$$

Thus we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s - s_n|}{m \cdot |s|},$$

and it is clear how we should proceed with our proof (according to Ross).

Proof. Let $\epsilon > 0$. By example 6 in the Textbook Examples section, there exists $m > 0$ with $|s_n| \geq m$ for all n. Because $\lim s_n = s$, there exists N with

$$n > N \implies |s - s_n| < \epsilon \cdot m|s|.$$

Then $n > N$ gives

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n \cdot s|} \leq \frac{|s - s_n|}{m \cdot |s|} < \epsilon.$$

□

This is the proof we do in class. Consider

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{|s_n \cdot s|}$$

Should bound $|s_n \cdot s|$ below by a constant.

Proof. Since $s_n \neq 0, s \neq 0$, we can find $M > 0$ with $|s_n| > M, \forall n$. Then take N with

$$|s - s_n| < |s|M\epsilon, \forall n > N.$$

Then,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} < \frac{|s - s_n|}{|s|M} < \frac{|s|M\epsilon}{|s|M} = \epsilon$$

□

Theorem 2.6. Suppose (s_n) converges to s, and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0, \forall n$, then $(\frac{t_n}{s_n})$ converes to $\frac{t}{s}$.

Proof. By Theorem 3.5 (Ross Lemma 9.5), the sequence $(1/s_n)$ converges to $1/s$, so theorem 3.4 (Ross 9.4) gives us

$$\lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} \cdot t_n = \frac{1}{s} \cdot t = \frac{t}{s},$$

as required. □

3 Limits of Basic Examples

- $\lim_{n\to\infty}\left(\frac{1}{n^p}\right) = 0$, for $p > 0$.
- $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.
- $\lim_{n\to\infty}(n^{1/n}) = 1$.
- $\lim_{n\to\infty}(a^{1/n}) = 1$, for $a > 0$.

Example: Prove that $\lim s_n = \frac{1}{4}$, where

$$s_n := \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}$$

Proof. Consider $s_n = \frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}}$. By 9.7, we know

$$\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3} \to 0$$

$$\implies \frac{6}{n}, \frac{7}{n^3}, \frac{3}{n^2}, \frac{-4}{n^3} \to 0$$

So $1 + \frac{6}{n} + \frac{7}{n^3} \to 1$, and $4 + \frac{3}{n^2} - \frac{4}{n^3} \to 4$ by 9.3. So $s_n \to \frac{1}{4}$ by 9.6. □

Lecture ends here.