Math 104, Summer 2019

Lecture 3, Wednesday 6/26/2019

CLASS ANNOUNCEMENTS: The standard of rigor expected out of a proof correlates with the pedagogical ideas behind the proofs. That is, higher-level results of proofs can use higher level reasoning. Remember that Homework 1 is due in class tomorrow, Thursday.

1 Review

Definition: Archimedean Property -

If $a > 0, b > 0$, then there is some $n \in \mathbb{N}$ with $na > b$.

Definition: Denseness of $\mathbb{Q}$ -

If $a, b \in \mathbb{R}$ and $a < b$ then $\exists r \in \mathbb{Q}$ with

$$a < r < b.$$

(i.e. $\exists m, n \in \mathbb{Z}$ with $a < \frac{m}{n} < b$ which is also $na < m < nb$. So we'd need to choose some n large enough where there is some $m \in \mathbb{Z}$ between the two.)

2 The Symbols $+\infty$ and $-\infty$

Remark:

- **not** numbers
- generally do not manipulate them algebraically; exceptions:

$$\infty + \infty = \infty; (-\infty) + (-\infty) = (-\infty); -(\infty) = (-\infty)$$

Definition: $\pm\infty$

If $A \subseteq \mathbb{R}$, we say $\sup A = \infty$ if A is not bounded above, and $\inf A = -\infty$ if A is not bounded below.

Example:

$$(-\infty, a] := \{x \in \mathbb{R} | x \leq a]\}$$
$$(-\infty, \infty) = \mathbb{R}$$

$$\sup A + \sup B = \sup(A + B)$$

$A, B \subseteq \mathbb{R}$, $(A+B) := \{a + b | a \in A, b \in B\}$. If A or B is allowed not to be bounded above (or both), the formula asserts that $A + B$ is **not** bounded above.

Proof. Suppose WLOG A is not bounded above. Let $M > 0$. Pick $b \in B$ arbitrarily. Then $\exists_{a \in A}(a > M - b)$. Then $a + b > M$, so that $A + B$ is not bounded above, so $\sup(A + B) = \infty$. □

3 Dedekind Cut

Definition: Dedekind Cut -

A subset $\alpha \subset \mathbb{Q}$ is called a **Dedekind cut** if it has the properties:

(i) $\alpha \neq \mathbb{Q}, \alpha \neq \{\}$

(ii) $(r \in \alpha, s \in \mathbb{Q}, s < r) \implies (s \in \alpha)$

(iii) α has no maximal element (no largest rational)

Remark: $\mathbb{R}$ is (now) defined as the set consisting of **all** Dedekind cuts. In other words, $\mathbb{R}$ is the set whose elements are the Dedekind cuts.

Thus elements of $\mathbb{R}$ (real numbers $x \in \mathbb{R}$) are **defined** as certain subsets of $\mathbb{Q}$.
Each rational $s \in \mathbb{Q}$ corresponds to the Dedekind cut

$$s^* := \{r \in \mathbb{Q} \mid r < s\}.$$

In this way, $\mathbb{Q}$ is regarded as a subset of $\mathbb{R}$; that is, $\mathbb{Q}$ is identified with the set:

$$\mathbb{Q} := \{s^* \mid s \in \mathbb{Q}\}.$$

3.1 Ordering of Dedekind Cuts

Recall that we talked about the "Well-Ordering Principle" (WOP) for rationals $\mathbb{Q}$ and $\mathbb{R}$ to allow us to say something like $-2 < \pi$. We briefly talked about how the induction axiom (Peano) is enough to prove the WOP, and vice-versa. But since Ross skips the precise construction of $\mathbb{R}$, we'll want some more rigorous way to define $\alpha < \beta$ for $\alpha, \beta \in \mathbb{R}$.

Definition: Ordering of Dedekind Cuts -

We define $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$ as subsets of $\mathbb{Q}$.

Definition: Addition of Dedekind Cuts -

We define $\alpha + \beta := \{r_1 + r_2 | r_1 \in \alpha_1, r_2 \in \beta\}$. We verify that if $r_1 \in \alpha, r_2 \in \beta$, then $s := r_1 + r_2 \in (\alpha + \beta)$

For more (and multiplication of Dedekind Cuts), see Section 1.6, page 30 Ross.

4 Sequences

A sequence is a function which assigns real numbers s_n to integers n (for $n \geq m \in \mathbb{Z}$), some starting index. We usually use $m := 1$ or $m := 0$.
To refer to sequences as whole objects, we write: $(S_n)_{n=m}^{\infty}$, $(S_n)_{n \in \mathbb{N}}$, and finally (S_n), if indexing is clear from context or unimportant.

Example:

$$\begin{aligned} S_n &:= n \\ S_n &:= \frac{1}{n^2} \\ S_n &:= (-1)^n \\ S_n &:= \cos\left(\frac{n\pi}{3}\right) \\ S_n &:= (n)^{1/n} \\ S_n &:= \left(1 + \frac{1}{n}\right)^n \end{aligned}$$

Questions: What happens to the values of a sequence as n goes arbitrarily large?

Definition: Convergence, Limit -

A sequence S_n of real numbers is said to **converge** to the real number s, provided that for **each** $\epsilon > 0$, there exists a number N with $(n > N \implies |s_n - s| < \epsilon)$. If s_n converges to s, we write

$$\lim_{n\to\infty} s_n = s \quad \text{or} \quad s_n \to s$$

If given ϵ, we have to generate or find N that leads to convergence. N as a function of ϵ would get larger for smaller ϵ.
Knowing s, and given ϵ, to show it converges, we show that we can find some number N for the index of the sequence term **past which** the value s_n (for $n > M$) is strictly within ϵ of s.
To show the limit does not exist, we need only pick some (single) ϵ for which we cannot find N that gives convergence.

Example: Showing a limit is unique.

Solution. Picture of two non-overlapping bands for supposed s, s' distinct limits. □

Remark: **Proving** convergence and **Using** convergence to prove something are very different.

Example: Finding a limit:

$$s_n = \frac{3n+1}{7n-4} = \frac{3 + \frac{1}{n}}{7 - \frac{4}{n}}$$

Later we'll formalize the intuition behind this. But for now, we are interested in that $\lim s_n = \frac{3}{7}$ means that:

$$\forall \epsilon > 0, \exists_N \left[n > N \implies \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon \right]$$

End of lecture.