Math 104, Summer 2019
Lecture 2, Tuesday 6/25/2019

In last lecture, we introduced $\mathbb{N}$ (via introducing Peano Axioms).

1 Axioms of $\mathbb{R}$

For all $a, b, c \in \mathbb{R}$,
A1. $a + (b + c) = (a + b) + c$; (assoc. of add)
A2. $a + b = b + a$ (commut. of add.)
A3. $a + 0 = a$ (exist. of add. ident. 0)
A4. For each a, $\exists -a$ with $a + (-a) = 0$. (add. inverse)
M1. $a(bc) = (ab)c$ (assoc. of mult.)
M2. $ab = ba$ (commut. of mult.)
M3. $a \cdot 1 = a$ (exist. of mult. identity 1)
M4. For each $a \neq 0, \exists a^{-1}$ with $aa^{-1} = 1$ (exist. of mult. inverse)
DL. $a(b + c) = ab + ac$ (distributive law)

The set $\mathbb{Q}$ has an order structure $\leq$ satisfying:

O1. Given a, b, either $a \leq b$ or $b \leq a$.
O2. $a \leq b, b \leq a \implies a = b$.
O3. $a \leq b, b \leq c \implies a \leq c$ (transitive law)
O4. $a \leq b \implies a + c \leq b + c$
O5. $a \leq b, 0 \leq c \implies ac \leq bc$

From the axioms, we have all the following implications proven in HW1.

Theorem 3.1, Ross p.15 (Consequences of Field Properties)
(i) $a + c = b + c \implies a = b$;
(ii) $a \cdot 0 = 0, \quad \forall a$;
(iii) $(-a)b = -ab \quad \forall a, b$;
(iv) $(-a)(-b) = ab \quad \forall a, b$;
(v) $ac = bc, c \neq 0 \implies a = b$;
(vi) $ab = 0 \implies [(a = 0) \cup (b = 0)], \quad \forall a, b, c \in \mathbb{R}$.

Theorem 3.2, Ross p.16 (Consequences of an Ordered Field) For all $a, b, c \in \mathbb{R}$,
(i) $a \leq b \implies -b \leq -a$;
(ii) $a \leq b, c \leq 0 \implies bc \leq ac$;
(iii) $0 \leq a, 0 \leq b \implies 0 \leq ab$;
(iv) $0 \leq a^2, \quad \forall a$;
(v) $0 < 1$;
(vi) $0 < a \implies 0 < a^{-1}$;
(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$;

2 Absolute Value

Definition: Absolute Value - Let $a \in \mathbb{F}$. Then we define:

$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Properties (Theorems) of Absolute Values:

(i) $|a| \geq 0$
(ii) $|a \cdot b| = |a| \cdot |b|$
(iii) $|a + b| \leq |a| + |b|$

To prove these, we can do case-work (and induction to generalize).

Definition: Distance -

$$\begin{aligned} \text{dist}(a, b) &:= |a - b| \\ \text{dist}(a, c) &\leq \text{dist}(a, b) + \text{dist}(b, c) \\ |a - b + b - c| &\leq |a - b| + |b - c| \end{aligned}$$

Exercise: Show that $\big||a| - |b|\big| \leq |a - b|$.

Solution. We split into cases: $a < b < 0$: reason
$b < a < 0$: reason
$a < 0 < b$: reason
$b < 0 < a$: reason
$0 < a < b$: This is trivially true.
$0 < b < a$: This is trivially true. □

Note: Max gave a really quick verbal casework proof, just with if - then.

What distinguishes $\mathbb{R}$ from $\mathbb{Q}$? To start, we notice there are 'gaps' in $\mathbb{Q}$, like transcendental numbers π which are not even algebraic numbers.

Dedekind Completeness Axiom:

Let S be a nonempty subset of $\mathbb{R}$.
If S contains a largest element s_0, then we call s_0 the **maximum** of S.

$$s_0 := \max S$$

If S contains a least element s_1, then we call s_1 the **minimum** of S.

$$s_0 := \min S$$

Example:

(a) finite sets $\max\{1, 2, \ldots, 5\} = 5, \min\{1, 2, \ldots, 5\} = 1$

(b) closed/open intervals: $[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$, $(a, b) := \{x \in \mathbb{R} | a < x < b\}$ (and $[a, b), (a, b]$. The open intervals have no max/min on their open bound.

3 Bounded Sets

Definition: Bounded set -

If S is a nonempty subset of $\mathbb{R}$ and M is a number with $s \leq M, \quad \forall s \in S$, then M is called an **upper bound** of S; and S is **bounded above**.
If S is a nonempty subset of $\mathbb{R}$ and M is a number with $M \leq s, \quad \forall s \in S$, then M is called an **lower bound** of S; and S is **bounded below**.

Notice that $\mathbb{Z}, \mathbb{Q}$ have no max or min, and are unbounded in either direction. $\mathbb{N}$ has no max and is not bounded above but has min 1.

Definition: Suprema, Infima (supremum, infimum) -

If $S \subset \mathbb{R}$ is nonempty and bounded *above*, and if S has a *least upper bound* (the set of upper bounds has a minimum) s, we call s the **supremum** of S and write

$$s := \sup S$$

If $S \subset \mathbb{R}$ is nonempty and bounded *below*, and if S has a *greatest lower bound* (the set of lower bounds has a maximum) s, we call s the **infimum** of S and write

$$s := \inf S$$

It turns out that all sets S in $\mathbb{R}$, if bounded above, has a supremum, and if bounded below, has an infimum. This provides a basis for the Completeness Axiom.

Example: $S := \{x \in \mathbb{Q} | x^2 \leq 2\}$ has no supremum in $\mathbb{Q}$, but $\sup S = \sqrt{2}$ in $\mathbb{R}$.

The Completeness Axiom:

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum (within $\mathbb{R}$).

Corollary:

If $S \in \mathbb{R}$ is nonempty and bounded below, then $\inf S$ exists in $\mathbb{R}$.

Theoretically, when we construct $\mathbb{R}$ to have certain characteristics, the completeness axiom follows. However we can state it here as an axiom to sort of "take it for granted".

The claim about inf can be **derived** from this Completeness axiom about sup.

Key Insight:

$$\inf S = -\sup(-S)$$

where

$$-S := \{-x | x \in S\}.$$

Remark: The axioms that we used for $\mathbb{Q}$ are not enough to **uniquely define** $\mathbb{Q}$. However, with the ordered field axioms and the completeness axiom, $\mathbb{R}$ is the **only** ordered field (and $\mathbb{R}$ **is uniquely defined**).

Definition: Archimedean Property -

If $a > 0, b > 0$, then there is some $n \in \mathbb{N}$ with $na > b$.

To prove this in $\mathbb{R}$, we use the completeness axiom. Essentially we use contradiction, that if no such n exists, then b is an upper bound (UB) for $S := \{na\}$. (If we cannot find some n, then b is larger than every na.)

Because b is an upper-bound and $S \subset \mathbb{R}$, $s_0 = \sup S$ exists.

Then $s_0 - a < n_0 a$ for some n_0. Then $s_0 < (n_0 + 1)a$, and $(n_0 + 1)a$ is larger than s_0, a contradiction.

Definition: Denseness of $\mathbb{Q}$ -

If $a, b \in \mathbb{R}$ and $a < b$ then $\exists r \in \mathbb{Q}$ with

$$a < r < b.$$

(i.e. $\exists m, n \in \mathbb{Z}$ with $a < \frac{m}{n} < b$ which is also $na < m < nb$. So we'd need to choose some n large enough where there is some $m \in \mathbb{Z}$ between the two.)

Basically, in between any two real numbers, we can find some rational number in between.

The proof of this follows more or less from the archimedian property as follows:

Pseudo-proof: - archimedean property tells us we can find some n with $n(b - a) > 1$.

Then with the choice n that gives $m \in \mathbb{Z}$ for $na < m < nb$, we can proceed.

Remark: Today's lecture ends here. Section 5 will be covered next time, as it's short.